

# TRACE SCALING AUTOMORPHISMS OF $\mathcal{W} \otimes \mathbb{K}$

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ABSTRACT. We classify trace scaling automorphisms of  $\mathcal{W} \otimes \mathbb{K}$  up to outer conjugacy, where  $\mathcal{W}$  is a certain simple separable nuclear stably projectionless  $C^*$ -algebra having trivial  $K$ -groups. Also, we show that all automorphisms of  $\mathcal{W}$  with the Rohlin property are outer conjugate to each other.

## 1. INTRODUCTION

Let  $\mathcal{W}$  be the Razak-Jacelon algebra studied in [20], which is a certain simple nuclear stably projectionless  $C^*$ -algebra having trivial  $K$ -groups and a unique tracial state and no unbounded traces. We may regard  $\mathcal{W}$  as a stably finite analogue of the Cuntz algebra  $\mathcal{O}_2$ . Note that a  $C^*$ -algebra  $A$  is said to be *stably projectionless* if  $A \otimes \mathbb{K}$  has no non-zero projections, where  $\mathbb{K}$  is the  $C^*$ -algebra of compact operators on an infinite-dimensional separable Hilbert space. In particular, every stably projectionless  $C^*$ -algebra is non-unital. We refer the reader to [12] and [15] for remarkable progress in the classification of such  $C^*$ -algebras.

In this paper, we study trace scaling automorphisms of  $\mathcal{W} \otimes \mathbb{K}$  and show that these automorphisms are outer conjugate if and only if the scaling factors coincide. This classification can be regarded as an analogous result of Connes' classification [5] of trace scaling automorphisms of the AFD factor of type  $II_\infty$ . In the case of  $C^*$ -algebras, Elliott, Evans and Kishimoto [10] classified trace scaling automorphisms of stable UHF algebras. Moreover, Evans and Kishimoto [13] classified trace scaling automorphisms of stable AF algebras with totally ordered  $K_0$ -groups. (See also [3].) More generally, the study of group actions on operator algebras is one of the most fundamental subjects and has a long history in the theory of operator algebras. We refer the reader to [19] and the references given there for this subject. We recall some other classification results of automorphisms of  $C^*$ -algebras. Kishimoto [24] showed that if  $\alpha$  and  $\beta$  are automorphisms of a UHF algebra such that  $\alpha^m$  and  $\beta^m$  are strongly outer for any  $m \in \mathbb{Z} \setminus \{0\}$ , then  $\alpha$  and  $\beta$  are outer conjugate. Moreover, Kishimoto classified a large class of automorphisms of certain AT algebras in [26] and [27]. Matui [34] generalized this result to certain simple AH algebras. Nakamura [40] completely classified aperiodic automorphisms of Kirchberg algebras. Sato [49] showed that if  $\alpha$  and  $\beta$  are automorphisms of the Jiang-Su algebra  $\mathcal{Z}$  such that  $\alpha^m$  and  $\beta^m$  are strongly outer for any  $m \in \mathbb{Z} \setminus \{0\}$ , then  $\alpha$  and  $\beta$  are outer conjugate. Note that it is important to consider the Rohlin property for classifying automorphisms of operator algebras.

If  $A$  is stably projectionless, then the central sequence  $C^*$ -algebra  $A_\omega$  of  $A$  is also stably projectionless. Hence  $A_\omega$  is not very useful for our purpose. In this paper, Kirchberg's central sequence  $C^*$ -algebra [21] plays a central role. Kirchberg's central sequence  $C^*$ -algebra  $F(A)$  is defined as the quotient  $C^*$ -algebra of  $A_\omega$  by the annihilator of  $A$ .

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This paper is organized as follows: In Section 2, we collect notations and some results. In Section 3, we review some results in [12] and reformulate for our purpose. Note that our arguments are essentially based on Elliott and Niu's arguments. In Section 4, we study properties of  $F(\mathcal{W})$ . We show that  $F(\mathcal{W})$  has many projections (Proposition 4.2). This is an easy corollary of Razak's classification theorem [44] and Matui and Sato's result in [38]. But this property enables us to deal with  $F(\mathcal{W})$  like a  $C^*$ -algebra of real rank zero. In Section 5, we obtain a homotopy type theorem for unitaries in  $F(\mathcal{W})$  by classifying certain unitaries in  $F(\mathcal{W})$  up to unitary equivalence (Theorem 5.7, Theorem 5.3). Some arguments in this section are motivated by arguments in [35, Section 4] (see also [31]). In Section 6, we introduce the Rohlin property for automorphisms of  $\sigma$ -unital  $C^*$ -algebras and show that every trace scaling automorphism of  $\mathcal{W} \otimes \mathbb{K}$  has the Rohlin property (Theorem 6.4). Moreover, we show that if  $\alpha$  is an automorphism of  $\mathcal{W}$  such that  $\alpha^m$  is strongly outer for any  $m \in \mathbb{Z} \setminus \{0\}$ , then  $\alpha$  has the Rohlin property (Theorem 6.7). In Section 7, we obtain a classification theorem of trace scaling automorphisms of  $\mathcal{W} \otimes \mathbb{K}$  by using the Bratteli-Elliott-Evans-Kishimoto intertwining argument (Theorem 7.3). Moreover, we show that if  $\alpha$  and  $\beta$  are automorphisms of  $\mathcal{W}$  such that  $\alpha^m$  and  $\beta^m$  are strongly outer for any  $m \in \mathbb{Z} \setminus \{0\}$ , then  $\alpha$  and  $\beta$  are outer conjugate (Theorem 7.4).

## 2. PRELIMINARIES

In this section we shall collect notations and some results. We refer the reader to [1] and [43] for basic facts of operator algebras.

**2.1. Notation.** We say that a  $C^*$ -algebra  $A$  is  $\sigma$ -unital if  $A$  has a countable approximate unit. Note that if  $A$  is separable, then  $A$  is  $\sigma$ -unital. If  $A$  is  $\sigma$ -unital, then there exists a positive element  $s \in A$  such that  $\{s^{\frac{1}{n}}\}_{n \in \mathbb{N}}$  is an approximate unit. Such a positive element  $s$  is called *strictly positive* in  $A$ . We denote by  $A^\sim$  the unitization algebra of  $A$ . The *multiplier algebra*, denoted by  $M(A)$ , of  $A$  is the largest unital  $C^*$ -algebra that contains  $A$  as an essential ideal. If  $\alpha$  is an automorphism of  $A$ , then  $\alpha$  extends uniquely to an automorphism of  $M(A)$ . We denote it by the same symbol  $\alpha$  for simplicity.

For a unitary element  $u$  in  $M(A)$ , define an automorphism  $\text{Ad}(u)$  of  $A$  by  $\text{Ad}(u)(x) = uxu^*$  for  $x \in A$ . Such an automorphism is called an *inner automorphism*. Let  $\text{Aut}(A)$  denote the automorphism group of  $A$ , which is equipped with the topology of pointwise norm convergence. An automorphism  $\alpha$  is said to be *approximately inner* if  $\alpha$  is in the closure of the inner automorphism group. We say that two automorphisms  $\alpha$  and  $\beta$  are *approximately unitarily equivalent* if  $\alpha \circ \beta^{-1}$  is approximately inner, and are *outer conjugate* if there exist an automorphism  $\gamma$  of  $A$  and a unitary element  $u$  in  $M(A)$  such that

$$\alpha = \text{Ad}(u) \circ \gamma \circ \beta \circ \gamma^{-1}.$$

Let  $F$  be a subset of  $A$  and  $\varepsilon > 0$ . A completely positive (c.p.) map  $\varphi : A \rightarrow B$  is said to be  $(F, \varepsilon)$ -multiplicative if

$$\|\varphi(xy) - \varphi(x)\varphi(y)\| < \varepsilon$$

for any  $x, y \in F$ . For c.p. maps  $\varphi, \psi : A \rightarrow B$ , we write  $\varphi \sim_{F, \varepsilon} \psi$  if there exists a unitary element  $u \in B^\sim$  such that

$$\|\varphi(x) - u\psi(x)u^*\| < \varepsilon$$

for any  $x \in F$ .

We denote by  $A_+$  the set of positive elements in  $A$  and by  $A_{+,1}$  the set of positive contractions in  $A$ . A *trace* on  $A$  is a map  $\tau$  of  $A_+$  to  $[0, \infty]$  such that

$\tau(\lambda a) = \lambda \tau(a)$ ,  $\tau(a+b) = \tau(a) + \tau(b)$  and  $\tau(x^*x) = \tau(xx^*)$  for any  $a, b \in A_+$ ,  $\lambda \geq 0$  and  $x \in A$ . For a trace  $\tau$  on  $A$ , let  $\mathfrak{M}_\tau$  be a linear span of  $\{a \in A_+ \mid \tau(a) < \infty\}$  and  $\mathfrak{N}_\tau := \{x \in A \mid \tau(x^*x) < \infty\}$ . Then  $\mathfrak{M}_\tau$  and  $\mathfrak{N}_\tau$  are ideals of  $A$  and  $\tau$  can be uniquely extended to a positive linear functional on  $\mathfrak{M}_\tau$ . A *tracial state* is a trace which is a state. We say that  $\tau$  is *densely defined* if  $\mathfrak{M}_\tau$  is dense in  $A$ , and is *lower semicontinuous* if  $\{a \in A_+ \mid \tau(a) \leq r\}$  is closed for any  $r \in \mathbb{R}_+$ . Let  $T(A)$  denote the set of densely defined lower semicontinuous traces on  $A$  and  $T_1(A)$  the set of tracial states on  $A$ . For  $\tau \in T(A)$ , put  $d_\tau(a) = \lim_{n \rightarrow \infty} \tau(a^{\frac{1}{n}})$  for  $a \in A_+$ . Then  $d_\tau$  is a dimension function. We denote by  $(\pi_\tau, H_\tau)$  the Gelfand-Naimark-Segal (GNS) representation of  $\tau \in T(A)$ . Note that  $H_\tau$  is the completion of the pre-Hilbert space  $\mathfrak{N}_\tau$  with a pre-inner product  $\langle \widehat{x}, \widehat{y} \rangle = \tau(y^*x)$  for  $x, y \in \mathfrak{N}_\tau$ . The norm on  $H_\tau$  is denoted by  $\|\cdot\|_2$ . Let  $\tau$  be a lower semicontinuous densely defined trace on a  $\sigma$ -unital  $C^*$ -algebra  $A$ . We denote by  $\text{Ped}(A)$  the Pedersen ideal of  $A$ , which is a minimal dense ideal of  $A$ . Note that  $\text{Ped}(A)$  is contained in  $\mathfrak{N}_\tau$  because  $\mathfrak{N}_\tau$  is a dense ideal in  $A$ . There exists an approximate unit  $\{h_n\}_{n \in \mathbb{N}}$  for  $A$  contained in  $\text{Ped}(A)$ . It is easy to see that  $\{\widehat{h_n x} \mid n \in \mathbb{N}, x \in \mathfrak{N}_\tau\}$  is dense in  $H_\tau$ . Indeed, the lower semicontinuity of  $\tau$  implies that for any  $x \in \mathfrak{N}_\tau$ , we have

$$\|\widehat{x} - \widehat{h_n x}\|_2 = \tau(x^*(1 - h_n)^2 x)^{\frac{1}{2}} \leq \tau(x^*(1 - h_n)x)^{\frac{1}{2}} \rightarrow 0$$

as  $n \rightarrow \infty$ . If  $\alpha$  is an automorphism of  $A$  such that  $\tau \circ \alpha = \lambda \tau$  for some  $\lambda \in \mathbb{R}_+^\times$ , then  $\alpha$  can be uniquely extended to an automorphism  $\tilde{\alpha}$  of  $\pi_\tau(A)''$ .

For  $x, y \in A$ , we write  $[x, y]$  to mean the commutator  $xy - yx$ . We denote by  $\mathbb{K}$  and  $M_{n^\infty}$  for  $n \in \mathbb{N}$  the  $C^*$ -algebra of compact operators on an infinite-dimensional separable Hilbert space and the uniformly hyperfinite (UHF) algebra of type  $n^\infty$ , respectively. Let  $\text{Tr}_n$  for  $n \in \mathbb{N}$  denote the (unnormalized) usual trace on  $M_n(\mathbb{C})$  and  $\text{Tr}$  denote the usual trace on  $\mathbb{K}$ .

**2.2. Kirchberg's central sequence  $C^*$ -algebras.** We shall recall some properties of Kirchberg's central sequence  $C^*$ -algebras in [21] (see also [41, Section 5]). Let  $\omega$  be a free ultrafilter on  $\mathbb{N}$ . For a  $\sigma$ -unital  $C^*$ -algebra  $A$ , set

$$c_\omega(A) := \{(x_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n \rightarrow \omega} \|x_n\| = 0\}, \quad A^\omega := \ell^\infty(\mathbb{N}, A)/c_\omega(A).$$

Let  $B$  be a  $C^*$ -subalgebra of  $A$ . We identify  $A$  and  $B$  with the  $C^*$ -subalgebras of  $A^\omega$  consisting of equivalence classes of constant sequences. Put

$$A_\omega := A^\omega \cap A', \quad \text{Ann}(B, A^\omega) := \{(x_n)_n \in A^\omega \cap B' \mid (x_n)_n b = 0 \text{ for any } b \in B\}.$$

Then  $\text{Ann}(B, A^\omega)$  is a closed ideal of  $A^\omega \cap B'$ , and define

$$F(A) := A_\omega / \text{Ann}(A, A^\omega).$$

We call  $F(A)$  the *central sequence  $C^*$ -algebra* of  $A$ . A sequence  $(x_n)_n$  is said to be *central* if  $\lim_{n \rightarrow \omega} \|[x_n, x]\| = 0$  for all  $x \in A$ . A central sequence is a representative of an element in  $A_\omega$ . Since  $A$  is  $\sigma$ -unital,  $A$  has a countable approximate unit  $\{h_n\}_{n \in \mathbb{N}}$ . It is easy to see that  $[(h_n)_n]$  is a unit in  $F(A)$ . If  $A$  is unital, then  $F(A) = A_\omega$ . Note that  $F(A)$  is isomorphic to  $M(A)^\omega \cap A' / \text{Ann}(A, M(A)^\omega)$  and  $A_\omega / \text{Ann}(A, (A^\sim)^\omega)$ . Let  $h$  be a full positive element in  $A$ , and define a map  $\eta$  from  $F(A)$  to  $F(\overline{hAh})$  by  $\eta([(x_n)_n]) = [(h^{\frac{1}{n}} x_n h^{\frac{1}{n}})_n]$ . Then  $\eta$  is an isomorphism from  $F(A)$  onto  $F(\overline{hAh})$ . In particular,  $F(A \otimes \mathbb{K})$  is isomorphic to  $F(A)$ . If  $\alpha$  is an automorphism of  $A$ ,  $\alpha$  induces natural automorphisms of  $A^\omega$ ,  $A_\omega$  and  $F(A)$ . We denote them by the same symbol  $\alpha$  for simplicity.

There exists a natural homomorphism  $\rho$  from  $F(A) \otimes_{\max} A$  to  $A^\omega$  such that

$$\rho([(x_n)_n] \otimes x) = (x_n x)_n$$

for any  $[(x_n)_n] \in F(A)$  and  $x \in A$ . For a projection  $p$  in  $F(A)$ , let  $A_p^\omega$  be a hereditary subalgebra of  $A^\omega$  generated by  $\rho(pF(A)p \otimes_{\max} A)$ . It can be easily checked that

$$A_p^\omega = \overline{\rho(p \otimes s)A^\omega \rho(p \otimes s)}$$

where  $s$  is a strictly positive element in  $A$ .

Since  $\omega$  is a (free) ultrafilter, for any  $\tau \in T_1(A)$ , we can define a tracial state  $\tau_\omega$  on  $A^\omega$  by  $\tau_\omega((a_n)_n) = \lim_{n \rightarrow \omega} \tau(a_n)$  for any  $(a_n)_n \in A^\omega$ . We shall show that  $\tau_\omega$  is well-defined on  $F(A)$ .

**Proposition 2.1.** Let  $A$  be a  $\sigma$ -unital  $C^*$ -algebra, and let  $\tau$  be a tracial state on  $A$ . Define  $\tau_\omega : F(A) \rightarrow \mathbb{C}$  by

$$\tau_\omega([(x_n)_n]) = \lim_{n \rightarrow \omega} \tau(x_n)$$

for any  $[(x_n)_n] \in F(A)$ . Then  $\tau_\omega$  is well-defined. In particular,  $\tau_\omega$  is a tracial state on  $F(A)$ .

*Proof.* It suffices to show that if  $(x_n)_n \in \text{Ann}(A, A^\omega)$ , then  $\lim_{n \rightarrow \omega} \tau(x_n) = 0$ . We may assume that  $\|x_n\| \leq 1$  for any  $n \in \mathbb{N}$ . Let  $\{h_n\}_{n \in \mathbb{N}}$  be an approximate unit for  $A$  and  $\varepsilon > 0$ . There exists a natural number  $N$  such that

$$|1 - \tau(h_N)| < \frac{\varepsilon}{2}$$

because  $\lim_{n \rightarrow \infty} \tau(h_n) = 1$ . Since  $\lim_{n \rightarrow \omega} \|x_n h_N\| = 0$ , there exists  $X \in \omega$  such that

$$|\tau(x_n h_N)| < \frac{\varepsilon}{2}$$

for any  $n \in X$ . Hence we have

$$\begin{aligned} |\tau(x_n)| &\leq |\tau(x_n) - \tau(x_n h_N)| + |\tau(x_n h_N)| < |\tau((1 - h_N)^{\frac{1}{2}} x_n (1 - h_N)^{\frac{1}{2}})| + \frac{\varepsilon}{2} \\ &\leq |1 - \tau(h_N)| + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

for any  $n \in X$ . Therefore  $\lim_{n \rightarrow \omega} \tau(x_n) = 0$ .  $\square$

For a semifinite von Neumann algebra  $M$  with separable predual, set

$$\mathcal{C}_\omega(M) := \{(x_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, M) \mid [x_n, y] \rightarrow 0 \text{ } * \text{-strongly as } n \rightarrow \omega \text{ for any } y \in M\}$$

and

$$\mathcal{T}_\omega(M) := \{(x_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, M) \mid x_n \rightarrow 0 \text{ } * \text{-strongly as } n \rightarrow \omega\}.$$

Then  $\mathcal{C}_\omega(M)$  is a  $C^*$ -subalgebra of  $\ell^\infty(\mathbb{N}, M)$  and  $\mathcal{T}_\omega(M)$  is a closed ideal of  $\mathcal{C}_\omega(M)$ . Define

$$M_\omega := \mathcal{C}_\omega(M) / \mathcal{T}_\omega(M).$$

Note that  $M_\omega$  coincides with the asymptotic centralizer of  $M$  in [4], and hence  $M_\omega$  is a finite von Neumann algebra. (Indeed, [51, Lemma XIV.3.4] and some arguments on semifinite von Neumann algebras show this fact.) Let  $p$  be a projection in  $M$  with central support 1, and define a map  $\tilde{\eta}$  from  $M_\omega$  to  $(pMp)_\omega$  by  $\tilde{\eta}((x_n)_n) = (px_n p)_n$ . Then  $\tilde{\eta}$  is an isomorphism from  $M_\omega$  onto  $(pMp)_\omega$  by [4, Lemma 2.11] (see also [32, Lemma 2.8]) and [32, Proposition 2.10]. If  $\alpha$  is an automorphism of a semifinite von Neumann algebra  $M$ ,  $\alpha$  induces a natural automorphism of  $M_\omega$ . We denote it by the same symbol  $\alpha$  for simplicity. Note that if  $\tau$  is a bounded trace, then the following proposition is clear.

**Proposition 2.2.** Let  $A$  be a  $\sigma$ -unital  $C^*$ -algebra, and let  $\tau$  be a lower semicontinuous densely defined trace on  $A$ . Then the inclusion map from  $A$  to  $\pi_\tau(A)''$  induces a homomorphism  $\varrho_A$  from  $F(A)$  to  $\pi_\tau(A)''_\omega$ .

*Proof.* Let  $\{h_m\}_{m \in \mathbb{N}}$  be an approximate unit for  $A$  contained in  $\text{Ped}(A)$ . First, we shall show that if  $(x_n)_n \in \text{Ann}(A, A^\omega)$ , then  $(\pi_\tau(x_n))_n \in \mathcal{T}_\omega(\pi_\tau(A))''$ . For any  $m \in \mathbb{N}$  and  $x \in \mathfrak{N}_\tau$ , we have

$$\|\pi_\tau(x_n)\widehat{h_m x}\|_2 + \|\pi_\tau(x_n^*)\widehat{h_m x}\|_2 \leq \|x_n h_m\| \cdot \|\widehat{x}\|_2 + \|x_n^* h_m\| \cdot \|\widehat{x}\|_2 \rightarrow 0$$

as  $n \rightarrow \omega$ . Hence we see that  $(\pi_\tau(x_n))_n \in \mathcal{T}_\omega(\pi_\tau(A))''$ .

Let  $(x_n)_n$  be a central sequence of contractions in  $A$ . If we prove  $(\pi_\tau(x_n))_n \in \mathcal{C}_\omega(\pi_\tau(A))''$ , then we obtain the conclusion. Let  $\varepsilon > 0$  and  $y \in \pi_\tau(M)''$ . It suffices to show that for any  $m \in \mathbb{N}$  and  $x \in \mathfrak{N}_\tau$ , there exists  $X \in \omega$  such that

$$\|[\pi_\tau(x_n), y]\widehat{h_m x}\|_2 + \|[\pi_\tau(x_n), y]^*\widehat{h_m x}\|_2 < \varepsilon$$

for any  $n \in X$ . We may assume that  $y$  is a contraction and  $\widehat{x} \neq 0$ . (Note that we also have  $x \neq 0$ .) By the Kaplansky density theorem, there exists a contraction  $z \in A$  such that

$$\|(y - \pi_\tau(z))\widehat{h_m}\|_2 + \|(y^* - \pi_\tau(z^*))\widehat{h_m}\|_2 < \frac{\varepsilon}{12\|\widehat{x}\|}.$$

Since  $(x_n)_n$  is a central sequence in  $A$ , there exists  $X \in \omega$  such that

$$\|[x_n, z]\| < \frac{\varepsilon}{6\|\widehat{x}\|_2}, \quad \|[x_n, h_m]\| < \frac{\varepsilon}{12\|\widehat{x}\|_2}$$

for any  $n \in X$ . Then we have

$$\begin{aligned} \|[\pi_\tau(x_n), y]\widehat{h_m x}\|_2 &\leq \|(\pi_\tau(x_n)y - \pi_\tau(x_n z))\widehat{h_m x}\|_2 + \|(\pi_\tau(x_n z) - y\pi_\tau(x_n))\widehat{h_m x}\|_2 \\ &\leq \|x_n\| \cdot \|(y - \pi_\tau(z))\widehat{h_m}\|_2 \cdot \|x\| + \|(\pi_\tau(x_n z) - y\pi_\tau(x_n))\widehat{h_m x}\|_2 \\ &< \frac{\varepsilon}{12} + \|\pi_\tau([x_n, z])\widehat{h_m x}\|_2 + \|(\pi_\tau(zx_n) - y\pi_\tau(x_n))\widehat{h_m x}\|_2 \\ &\leq \frac{\varepsilon}{12} + \|[x_n, z]\| \cdot \|h_m\| \cdot \|\widehat{x}\|_2 + \|(\pi_\tau(zx_n) - y\pi_\tau(x_n))\widehat{h_m x}\|_2 \\ &< \frac{\varepsilon}{4} + \|\pi_\tau(z[x_n, h_m])\widehat{x}\|_2 + \|\pi_\tau(zh_m x_n)\widehat{x} - y\pi_\tau(x_n)\widehat{h_m x}\|_2 \\ &\leq \frac{\varepsilon}{4} + \|z\| \cdot \|[x_n, h_m]\| \cdot \|\widehat{x}\|_2 + \|(\pi_\tau(zh_m x_n) - y\pi_\tau(x_n h_m))\widehat{x}\|_2 \\ &< \frac{\varepsilon}{3} + \|\pi_\tau(zh_m x_n)\widehat{x} - y\pi_\tau(h_m x_n)\widehat{x}\|_2 + \|y\pi_\tau([h_m, x_n])\widehat{x}\|_2 \\ &\leq \frac{\varepsilon}{3} + \|(\pi_\tau(z) - y)\widehat{h_m}\|_2 \cdot \|x_n\| \cdot \|x\| + \|y\| \cdot \|[h_m, x_n]\| \cdot \|\widehat{x}\|_2 \\ &< \frac{\varepsilon}{2} \end{aligned}$$

for any  $n \in X$ . Similar arguments shows

$$\|[\pi_\tau(x_n), y]^*\widehat{h_m x}\|_2 < \frac{\varepsilon}{2}$$

for any  $n \in X$ . Therefore the proof is complete.  $\square$

If  $\tau$  is a bounded faithful trace on  $A$ , then  $\ker(\varrho_A) = \{x \in F(A) \mid \tau_\omega(x^*x) = 0\}$ . Moreover, the same proof as [23, Theorem 3.3] shows that  $\varrho_A$  is surjective. (See also [50, Lemma 2.1].) Using this fact, we show the following proposition.

**Proposition 2.3.** Let  $A$  be a  $\sigma$ -unital simple  $C^*$ -algebra, and let  $\tau$  be a lower semicontinuous densely defined trace on  $A$ . Then  $\varrho_A$  is surjective.

*Proof.* Let  $h$  be a non-zero positive element in  $\text{Ped}(A)$ . Then  $\overline{hAh}$  is a simple  $C^*$ -algebra with no unbounded traces by the same argument as in the proof of [41, Proposition 5.2]. In particular,  $\tau|_{\overline{hAh}}$  is a bounded faithful trace on  $\overline{hAh}$ . Let  $p$  be a support projection of  $\pi_\tau(h)$  in  $\pi_\tau(A)''$ . Note that  $\pi_\tau(h^{\frac{1}{n}})$  converges to  $p$  in the

strong\* topology as  $n \rightarrow \infty$ . Since  $h$  is full in  $A$ ,  $p$  has central support 1. Hence we have the following commutative diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{\varrho_A} & \pi_\tau(A)''_\omega \\ \downarrow \eta & & \downarrow \tilde{\eta} \\ F(\overline{hAh}) & \xrightarrow{\varrho_{\overline{hAh}}} & (p\pi_\tau(A)''p)_\omega \end{array}$$

where  $\eta$  and  $\tilde{\eta}$  are standard isomorphisms from  $F(A)$  onto  $F(\overline{hAh})$  and from  $\pi_\tau(A)''_\omega$  onto  $(p\pi_\tau(A)''p)_\omega$ , respectively. Since  $\varrho_{\overline{hAh}}$  is surjective, we see that  $\varrho_A$  is surjective.  $\square$

**Remark 2.4.** Let  $A$  be a  $\sigma$ -unital simple  $C^*$ -algebra, and let  $\tau$  be a lower semicontinuous densely defined trace on  $A$ . Then it can be easily checked that  $\ker(\varrho_A) = \{[(x_n)_n] \in F(A) \mid \lim_{n \rightarrow \omega} \|\pi_\tau(x_n)\hat{h}\|_2 = 0\}$  for some  $h \in \text{Ped}(A)_+ \setminus \{0\}$ . Also, we can define a tracial state  $\tau_\omega$  on  $A$  by  $\tau_\omega([(x_n)_n]) := \lim_{n \rightarrow \omega} \tau(x_n h) / \tau(h)$ . Note that if  $[(x_n)_n] \in \ker(\varrho_A)$ , then  $\tau_\omega([(x_n)_n]) = 0$ .

**2.3. Matui and Sato's result.** We shall remark that some arguments in [38] works for non-unital  $C^*$ -algebras. It is important to consider property (SI). For  $a, b \in A_+$ , we say that  $a$  is *Cuntz smaller than*  $b$ , written  $a \lesssim b$ , if there exists a sequence  $\{x_n\}_{n \in \mathbb{N}}$  of  $A$  such that  $\|x_n^* b x_n - a\| \rightarrow 0$ .

**Definition 2.5.** Let  $A$  be a  $C^*$ -algebra.

(1) Assume that  $T_1(A)$  is non-empty compact set. We say that  $A$  has *property (SI)* if for any central sequences  $(a_n)_n$  and  $(b_n)_n$  of positive contractions in  $A$  satisfying

$$\lim_{n \rightarrow \omega} \max_{\tau \in T_1(A)} \tau(a_n) = 0, \quad \inf_{m \in \mathbb{N}} \lim_{n \rightarrow \omega} \min_{\tau \in T_1(A)} \tau(b_n^m) > 0,$$

there exists a central sequence  $(s_n)_n$  in  $A$  such that

$$\lim_{n \rightarrow \omega} \|s_n^* s_n - a_n\| = 0, \quad \lim_{n \rightarrow \omega} \|b_n s_n - s_n\| = 0.$$

(2) Let  $S$  be a subset of  $T_1(A)$ . We say that  $A$  has *strict comparison* (respectively, *strict comparison with respect to*  $S$ ) if for any  $k \in \mathbb{N}$ ,  $a, b \in M_k(A)_+$  with  $d_{\tau \otimes \text{Tr}_k}(a) < d_{\tau \otimes \text{Tr}_k}(b)$  for any  $\tau \in T_1(A)$  (respectively, for any  $\tau \in S$ ) implies  $a \lesssim b$ .

Note that strict comparison in the definition above is different from almost unperforation of the Cuntz semigroup  $\text{Cu}(A)$ . Essentially same proofs as [38, Lemma 4.7] and [38, Proposition 4.8] show the following theorem. See also [39, Proposition 3.3] and [2, Theorem 4.1].

**Theorem 2.6.** (Matui-Sato)

Let  $A$  be a separable simple infinite-dimensional nuclear  $C^*$ -algebra with finitely many extremal tracial states and no unbounded traces. Assume that  $A$  has property (SI). Then:

- (i) For any tracial state  $\sigma$  on  $F(A)$ , there exists a tracial state  $\tau$  on  $A$  such that  $\sigma = \tau_\omega$ .
- (ii) If  $a$  and  $b$  are positive elements in  $F(A)$  satisfying  $d_{\tau_\omega}(a) < d_{\tau_\omega}(b)$  for any  $\tau \in T_1(A)$ , then there exists an element  $r \in F(A)$  such that  $r^* b r = a$ . Moreover,  $F(A)$  has strict comparison.

**2.4. Razak-Jacelon algebra.** Let  $\mathcal{W}$  be the Razak-Jacelon algebra studied in [20], which has trivial  $K$ -groups and a unique tracial state  $\tau$  and no unbounded traces. The Razak-Jacelon algebra  $\mathcal{W}$  is constructed as an inductive limit  $C^*$ -algebra of

Razak's building block in [44], that is,

$$A(n, m) = \left\{ f \in C([0, 1]) \otimes M_m(\mathbb{C}) \mid f(0) = \text{diag}(\overbrace{\tilde{c}, \dots, \tilde{c}}^k, 0_n), f(1) = \text{diag}(\overbrace{\tilde{c}, \dots, \tilde{c}}^{k+1}), \right. \\ \left. c \in M_n(\mathbb{C}) \right\},$$

where  $n$  and  $m$  are natural numbers with  $n|m$  and  $k := m/n - 1$ . Let  $\mathcal{O}_2$  denote the Cuntz algebra generated by 2 isometries  $S_1$  and  $S_2$ . For every  $\lambda_1, \lambda_2 \in \mathbb{R}$  there exists by universality a one-parameter automorphism group  $\alpha$  of  $\mathcal{O}_2$  given by  $\alpha_t(S_j) = e^{it\lambda_j} S_j$ . Kishimoto and Kumjian showed that if  $\lambda_1$  and  $\lambda_2$  are all non-zero, of the same sign and  $\lambda_1$  and  $\lambda_2$  generate  $\mathbb{R}$  as a closed subgroup, then  $\mathcal{O}_2 \rtimes_{\alpha} \mathbb{R}$  is a simple stably projectionless  $C^*$ -algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace in [29] and [30]. Moreover, Robert [45] showed that  $\mathcal{W} \otimes \mathbb{K}$  is isomorphic to  $\mathcal{O}_2 \rtimes_{\alpha} \mathbb{R}$  for some  $\lambda_1$  and  $\lambda_2$ . (See also [9].)

By the uniqueness of traces on  $\mathcal{W} \otimes \mathbb{K}$ , for any automorphism  $\alpha$  of  $\mathcal{W} \otimes \mathbb{K}$ , there exists a positive real number  $\lambda(\alpha)$  such that  $\tau \otimes \text{Tr} \circ \alpha = \lambda(\alpha) \tau \otimes \text{Tr}$ . We say that  $\alpha$  is a *trace scaling automorphism* if  $\lambda(\alpha) \neq 1$ . The following theorem is an immediate consequence of Razak's classification theorem (see also [45]).

**Theorem 2.7.** (Razak)

- (i) Let  $A$  be a simple unital approximately finite-dimensional (AF) algebra with a unique tracial state. Then  $A \otimes \mathcal{W}$  is isomorphic to  $\mathcal{W}$ .
- (ii) Every automorphism of  $\mathcal{W}$  is approximately inner.
- (iii) Let  $\alpha$  and  $\beta$  be automorphisms of  $\mathcal{W} \otimes \mathbb{K}$ . Then  $\alpha$  and  $\beta$  are approximately unitarily equivalent if and only if  $\lambda(\alpha) = \lambda(\beta)$ .
- (iv) For any  $\lambda \in \mathbb{R}_+^{\times}$ , there exists an automorphism  $\alpha$  of  $\mathcal{W} \otimes \mathbb{K}$  such that  $\lambda(\alpha) = \lambda$ .

Note that  $\mathcal{W}$  is  $\mathcal{Z}$ -stable by (i) in the theorem above. Hence  $\mathcal{W}$  has strict comparison and property (SI) (see [48], [37] and [41]).

### 3. STABLE UNIQUENESS THEOREM

In this section we shall recall some results in [12] and reformulate for our purpose. We denote by  $\mathbb{T}$  the unit circle in the complex plane. The following proposition is based on the results of [7], [8], [11] and [14].

**Proposition 3.1.** (cf. [12, Proposition 8.2])

Let  $A$  be a separable non-unital  $C^*$ -algebra and  $B$  a separable  $C^*$ -algebra, and let  $\sigma$  be a full homomorphism from  $A$  to  $B$ . Suppose that  $\varphi$  and  $\psi$  are nuclear homomorphisms from  $C(\mathbb{T}) \otimes A$  to  $B$  with  $[\varphi] = [\psi]$  in  $KK_{\text{nuc}}(C(\mathbb{T}) \otimes A, B)$ . Then for any finite subsets  $F_1 \subset C(\mathbb{T})$ ,  $F_2 \subset A$  and  $\varepsilon > 0$ , there exist  $m \in \mathbb{N}$ ,  $z_1, z_2, \dots, z_m \in \mathbb{T}$  and a unitary element  $u$  in  $M_{m^2+1}(B)^{\sim}$  such that

$$\begin{aligned} & \left\| u^* \left( \varphi(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k) \sigma(a) \oplus \dots \oplus \bigoplus_{k=1}^m f(z_k) \sigma(a)}^m \right) u \right. \\ & \quad \left. - \psi(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k) \sigma(a) \oplus \dots \oplus \bigoplus_{k=1}^m f(z_k) \sigma(a)}^m \right\| < \varepsilon \end{aligned}$$

for any  $f \in F_1$  and  $a \in F_2$ .

*Proof.* Choose a dense subset  $\{z_k \mid k \in \mathbb{N}\} \subset \mathbb{T}$ . Let  $\gamma$  be a homomorphism from  $A \otimes C(\mathbb{T})$  to  $M(B \otimes \mathbb{K} \otimes \mathbb{K})$  such that

$$\gamma(a \otimes f) = \sigma(a) \otimes \sum_{k=1}^{\infty} f(z_k) e_{kk} \otimes 1$$

for any  $a \in A$  and  $f \in C(\mathbb{T})$ , where  $\{e_{ij}\}_{i,j \in \mathbb{N}}$  is the standard matrix units of  $\mathbb{K}$ . Then we see that  $\gamma$  is purely large as an extension by [11, Theorem 17 (iii)]. The same arguments as in the proofs of [12, Lemma 8.1] and [12, Proposition 8.2] show that there exists a sequence  $\{u_n\}_{n \in \mathbb{N}}$  of unitaries in  $(B \otimes \mathbb{K})^\sim$  such that

$$\|u_n(\varphi(a \otimes f) \oplus \gamma(a \otimes f))u_n^* - \psi(a \otimes f) \oplus \gamma(a \otimes f)\| \rightarrow 0$$

as  $n \rightarrow \infty$  for any  $a \in A$  and  $f \in C(\mathbb{T})$ . For any  $m \in \mathbb{N}$ , let

$$e_m := 1 \oplus \sum_{k=1}^m e_{kk} \otimes \sum_{k=1}^m e_{kk} \in B^\sim \oplus B^\sim \otimes \mathbb{K} \otimes \mathbb{K} \subset B^\sim \otimes \mathbb{K}.$$

For any  $n \in \mathbb{N}$ , we have  $\|[e_m, u_n]\| \rightarrow 0$  as  $m \rightarrow \infty$ . Hence for sufficiently large  $m$ ,  $e_m u_n e_m$  is close to a unitary element in  $e_m(B \otimes \mathbb{K})^\sim e_m \cong M_{m^2+1}(B)^\sim$ .

Therefore choose a sufficient large  $n \in \mathbb{N}$ , and then choose a sufficiently large  $m \in \mathbb{N}$ , we can find a unitary element  $u$  in  $M_{m^2+1}(B)^\sim$  satisfying the conclusion of the proposition.  $\square$

In order to obtain a stable uniqueness theorem for  $(G, \delta)$ -multiplicative maps, we need to consider homomorphisms from  $A$  to  $\prod B_n / \bigoplus B_n$  for some  $C^*$ -algebras  $A$  and  $B_n$ . We can avoid the assumption of separability in the proposition above by Blackadar's technique (see [1, II.8.5] and [12, Lemma 8.4]). It is useful to consider the following for the fullness.

**Definition 3.2.** (cf. [12, Definition 8.7])

Let  $L$  be a map from  $A_{+,1} \setminus \{0\} \times (0, 1)$  to  $\mathbb{N}$  and  $N$  a map from  $A_{+,1} \setminus \{0\} \times (0, 1)$  to  $(0, \infty)$ . A homomorphism  $\varphi$  from  $A$  to  $B$  is said to be  $(L, N)$ -full if for any  $\varepsilon \in (0, 1)$ ,  $a \in A_{+,1} \setminus \{0\}$  and  $b \in B_{+,1}$ , there exist elements  $x_1, x_2, \dots, x_{L(a, \varepsilon)}$  in  $B$  such that

$$\|x_i\| \leq N(a, \varepsilon)$$

for any  $i = 1, 2, \dots, L(a, \varepsilon)$  and

$$\|b - \sum_{i=1}^{L(a, \varepsilon)} x_i \varphi(a) x_i^*\| < \varepsilon.$$

The following proposition is a variant of [12, Proposition 8.12]. For finite sets  $F_1$  and  $F_2$ , let  $F_1 \odot F_2 := \{a \otimes b \mid a \in F_1, b \in F_2\}$ .

**Proposition 3.3.** Let  $A$  be a separable non-unital nuclear  $C^*$ -algebra that is  $KK$ -equivalent to  $\{0\}$ , and let  $L : A_{+,1} \setminus \{0\} \times (0, 1) \rightarrow \mathbb{N}$  and  $N : A_{+,1} \setminus \{0\} \times (0, 1) \rightarrow (0, \infty)$  be maps. Then for any finite subsets  $F_1 \subset C(\mathbb{T})$ ,  $F_2 \subset A$  and  $\varepsilon > 0$ , there exist finite subsets  $G_1 \subset C(\mathbb{T})$ ,  $G_2 \subset A$ ,  $m \in \mathbb{N}$ , and  $\delta > 0$  such that the following hold. Let  $B$  be a  $C^*$ -algebra. For any contractive  $(G_1 \odot G_2, \delta)$ -multiplicative maps  $\varphi, \psi : C(\mathbb{T}) \otimes A \rightarrow B$  and an  $(L, N)$ -full homomorphism  $\sigma : A \rightarrow B$ , there exist a unitary element  $u$  in  $M_{m^2+1}(B)^\sim$  and  $z_1, z_2, \dots, z_m \in \mathbb{T}$  such that

$$\begin{aligned} & \left\| u^* \left( \varphi(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k) \sigma(a)}^m \oplus \cdots \oplus \overbrace{\bigoplus_{k=1}^m f(z_k) \sigma(a)}^m \right) u \right. \\ & \quad \left. - \psi(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k) \sigma(a)}^m \oplus \cdots \oplus \overbrace{\bigoplus_{k=1}^m f(z_k) \sigma(a)}^m \right\| < \varepsilon \end{aligned}$$

for any  $f \in F_1$  and  $a \in F_2$ .



*Proof.* Let finite subsets  $F_1 \subset C(\mathbb{T})$ ,  $F_2 \subset A$  and  $\varepsilon > 0$ . On the contrary, suppose that the proposition were false for  $F_1$ ,  $F_2$  and  $\varepsilon$ . Then for any  $n \in \mathbb{N}$ , there exist a  $C^*$ -algebra  $B_n$ , contractive c.p. maps  $\varphi_n, \psi_n : A \otimes C(\mathbb{T}) \rightarrow B_n$  and an  $(L, N)$ -full homomorphism  $\sigma_n : A \rightarrow B_n$  such that

$$\|\varphi_n(xy) - \varphi_n(x)\varphi_n(y)\| \rightarrow 0, \quad \|\psi_n(xy) - \psi_n(x)\psi_n(y)\| \rightarrow 0$$

as  $n \rightarrow \infty$  for any  $x, y \in A \otimes C(\mathbb{T})$  and there exist no unitaries in  $M_{n^2+1}(B_n)^\sim$  and  $n$  elements in  $\mathbb{T}$  satisfying the conclusion of the proposition.

Define homomorphisms  $\Phi$  and  $\Psi$  from  $A \otimes C(\mathbb{T})$  to  $\prod B_n / \bigoplus B_n$  by

$$\Phi(x) := (\varphi_n(x))_n, \quad \Psi(x) := (\psi_n(x))_n$$

for any  $x \in A \otimes C(\mathbb{T})$ , and define a homomorphism  $\Sigma$  from  $A$  to  $\prod B_n / \bigoplus B_n$  by

$$\Sigma(a) := (\sigma_n(a))_n$$

for any  $a \in A$ . Since  $\sigma_n$  is  $(L, N)$ -full for any  $n \in \mathbb{N}$ ,  $\Sigma$  is full in  $\prod B_n / \bigoplus B_n$ . By [12, Lemma 8.4], there exists a separable  $C^*$ -subalgebra  $B$  of  $\prod B_n / \bigoplus B_n$  such that

$$\Phi(A \otimes C(\mathbb{T})), \Psi(A \otimes C(\mathbb{T})), \Sigma(A) \subset B$$

and  $\Sigma$  is full in  $B$ .

Since  $A \otimes C(\mathbb{T})$  is non-unital, nuclear and  $KK$ -equivalent to  $\{0\}$ , it follows from Proposition 3.1 that there exist  $m \in \mathbb{N}$ ,  $z_1, z_2, \dots, z_m \in \mathbb{T}$  and a unitary element  $U$  in  $M_{m^2+1}(B)^\sim$  such that

$$\begin{aligned} & \|U^*(\Phi(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k)\Sigma(a)}^m \oplus \cdots \oplus \overbrace{\bigoplus_{k=1}^m f(z_k)\Sigma(a)}^m)U \\ & - \Psi(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k)\Sigma(a)}^m \oplus \cdots \oplus \overbrace{\bigoplus_{k=1}^m f(z_k)\Sigma(a)}^m) \| < \varepsilon \end{aligned}$$

for any  $f \in F_1$  and  $a \in F_2$ . It is easy to see that  $U$  can be lifted to a unitary element  $(u_n)_{n \in \mathbb{N}}$  in  $\prod M_{m^2+1}(B_n)^\sim$ . Note that we may assume  $u_n = b_n + 1$  for some  $b_n \in M_{m^2+1}(B_n)$ . For sufficiently large  $n \in \mathbb{N}$ , we have

$$\begin{aligned} & \|u_n^*(\varphi_n(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k)\sigma_n(a)}^m \oplus \cdots \oplus \overbrace{\bigoplus_{k=1}^m f(z_k)\sigma_n(a)}^m)u_n \\ & - \psi_n(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k)\sigma_n(a)}^m \oplus \cdots \oplus \overbrace{\bigoplus_{k=1}^m f(z_k)\sigma_n(a)}^m) \| < \varepsilon \end{aligned}$$

for any  $f \in F_1$  and  $a \in F_2$ . Take elements  $z_{m+1}, \dots, z_n \in \mathbb{T}$ . It can be easily checked that there exists a unitary element  $u$  in  $M_{n^2+1}(B_n)^\sim$  such that

$$\begin{aligned} & \|u^*(\varphi_n(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^n f(z_k)\sigma_n(a)}^n \oplus \cdots \oplus \overbrace{\bigoplus_{k=1}^n f(z_k)\sigma_n(a)}^n)u \\ & - \psi_n(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^n f(z_k)\sigma_n(a)}^n \oplus \cdots \oplus \overbrace{\bigoplus_{k=1}^n f(z_k)\sigma_n(a)}^n) \| < \varepsilon \end{aligned}$$

for any  $f \in F_1$  and  $a \in F_2$ . This is a contradiction. Therefore the proof is complete.  $\square$

The following lemma is an analogous lemma of [22, Lemma 2.2 (iii)].

**Lemma 3.4.** Let  $A$  be a  $C^*$ -algebra, and let  $L : A_{+,1} \setminus \{0\} \times (0, 1) \rightarrow \mathbb{N}$  and  $N : A_{+,1} \setminus \{0\} \times (0, 1) \rightarrow (0, \infty)$  be maps. Then there exist maps  $L' : A_{+,1} \setminus \{0\} \times (0, 1) \rightarrow \mathbb{N}$  and  $N' : A_{+,1} \setminus \{0\} \times (0, 1) \rightarrow (0, \infty)$  such that the following hold. Let  $B$  be a  $C^*$ -algebra, and let  $C$  be a hereditary subalgebra of  $B$ . If  $\sigma$  is a homomorphism from  $A$  to  $C \subset B$  such that for any  $\varepsilon \in (0, 1)$ ,  $a \in A_{+,1} \setminus \{0\}$  and  $b \in C_{+,1}$ , there exist elements  $x_1, x_2, \dots, x_{L(a,\varepsilon)}$  in  $B$  such that

$$\|x_i\| \leq N(a, \varepsilon)$$

for any  $i = 1, 2, \dots, L(a, \varepsilon)$  and

$$\|b - \sum_{i=1}^{L(a,\varepsilon)} x_i \varphi(a) x_i^*\| < \varepsilon,$$

then  $\sigma$  is  $(L', N')$ -full in  $C$ .

*Proof.* For any  $a \in A_{+,1} \setminus \{0\}$  and  $\varepsilon \in (0, 1)$ , let

$$L'(a, \varepsilon) := L(a^2, \varepsilon), \quad N'(a, \varepsilon) := N(a^2, \varepsilon).$$

Then  $L'$  and  $N'$  have the desired property. Indeed, let  $\sigma$  be a homomorphism from  $A$  to  $C \subset B$  satisfying the assumption above. For any  $b \in C_{+,1}$  and  $\varepsilon \in (0, 1)$ , there exist elements  $x_1, x_2, \dots, x_{L'(a,\varepsilon)}$  in  $B$  such that  $\|x_i\| \leq N'(a, \varepsilon)$  for any  $i = 1, 2, \dots, L'(a, \varepsilon)$  and

$$\|b^{\frac{1}{2}} - \sum_{i=1}^{L'(a,\varepsilon)} x_i \sigma(a^2) x_i^*\| < \varepsilon.$$

We have

$$\|b - \sum_{i=1}^{L'(a,\varepsilon)} b^{\frac{1}{4}} x_i \sigma(a^{\frac{1}{2}}) \sigma(a) \sigma(a^{\frac{1}{2}}) x_i^* b^{\frac{1}{4}}\| \leq \|b^{\frac{1}{2}} - \sum_{i=1}^{L'(a,\varepsilon)} x_i \sigma(a^2) x_i^*\| < \varepsilon.$$

Since  $C$  is a hereditary subalgebra of  $B$ , we see that  $b^{\frac{1}{4}} x_i \sigma(a^{\frac{1}{2}}) \in C$  for any  $i = 1, 2, \dots, L'(a, \varepsilon)$ . Therefore  $\sigma$  is  $(L', N')$ -full in  $C$ .  $\square$

Essentially the same proof as [12, Lemma 8.15] show the following lemma. Roughly speaking, this lemma says that if target algebras have strict comparison, then the  $(L, N)$ -fullness can be controlled by traces.

**Lemma 3.5.** (Elliott-Niu)

For any  $\varepsilon > 0$  and  $\delta > 0$ , there exist  $\ell(\delta) \in \mathbb{N}$  and  $n(\varepsilon) > 0$  such that the following hold. Let  $A$  be a  $C^*$ -algebra, and let  $S$  be a subset of  $T_1(A)$ . Assume that  $A$  has strict comparison with respect to  $S$ . If  $a$  and  $b$  are positive contractions in  $A$  such that

$$\tau(a) > d_\tau(b)\delta$$

for any  $\tau \in S$ , then there exist  $x_1, x_2, \dots, x_{\ell(\delta)} \in A$  such that

$$\|x_i\| \leq n(\varepsilon)$$

for any  $i = 1, 2, \dots, \ell(\delta)$  and

$$\|b - \sum_{i=1}^{\ell(\delta)} x_i a x_i^*\| < \varepsilon.$$

The following lemma is a variant of [34, Lemma 3.5].

**Lemma 3.6.** Let  $A$  be  $\sigma$ -unital  $C^*$ -algebra, and let  $\tau$  be an extremal tracial state on  $A$ . If  $p$  is a projection in  $F(A)$ , then

$$\tau_\omega(\rho(p \otimes x)) = \tau_\omega(p)\tau(x)$$

for any  $x \in A$ .

*Proof.* There exists a positive contraction  $(p_n)_n$  in  $\mathcal{W}_\omega$  such that  $p = [(p_n)_n]$ . Note that we have  $\rho(p \otimes x) = (p_n x)_n = (p_n x p_n)_n$ . For any  $x \in A$ , define  $\tau'(x) := \lim_{n \rightarrow \omega} \tau(p_n x)$ . Then  $\tau'$  is a trace on  $A$  and  $\tau' \leq \tau$ . Since  $\tau$  is an extremal tracial state on  $A$ , there exists  $\lambda \geq 0$  such that  $\tau' = \lambda\tau$ . Let  $\{h_m\}_{m \in \mathbb{N}}$  be an approximate unit for  $A$ . Then  $\lim_{m \rightarrow \infty} \tau(h_m) = 1$  because  $\tau$  is state. Hence  $\lambda = \lim_{m \rightarrow \infty} \tau'(h_m)$ . Similar arguments as in the proof of [41, Proposition 5.3] show

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \omega} \tau(p_n h_m) = \lim_{n \rightarrow \omega} \tau(p_n).$$

Therefore  $\lambda = \lim_{n \rightarrow \omega} \tau(p_n)$ . We obtain the conclusion.  $\square$

For a projection  $p$  in  $F(\mathcal{W})$ , define a homomorphism  $\sigma_p$  from  $\mathcal{W}$  to  $\mathcal{W}_p^\omega$  by

$$\sigma_p(x) := \rho(p \otimes x)$$

for any  $x \in \mathcal{W}$ .

**Proposition 3.7.** There exist maps  $L : \mathcal{W}_{+,1} \setminus \{0\} \times (0,1) \rightarrow \mathbb{N}$  and  $N : \mathcal{W}_{+,1} \setminus \{0\} \times (0,1) \rightarrow (0,\infty)$  such that the following hold. If  $p$  be a projection in  $F(\mathcal{W})$  such that  $\tau_\omega(p) > 0$  where  $\tau$  is the unique tracial state on  $\mathcal{W}$ , then  $\sigma_p$  is  $(L, N)$ -full.

*Proof.* For any  $\varepsilon > 0$  and  $\delta > 0$ , take  $\ell(\delta) \in \mathbb{N}$  and  $n(\varepsilon) > 0$  in Lemma 3.5. Define

$$L(a, \varepsilon) := \ell\left(\frac{\tau(a)}{2}\right), \quad N(a, \varepsilon) := n(\varepsilon)$$

for any  $a \in \mathcal{W}_{+,1} \setminus \{0\}$  and  $\varepsilon \in (0,1)$ . Note that  $\tau(a) > 0$  since  $\mathcal{W}$  is simple.

Let  $p \in F(\mathcal{W})$  be a projection in  $F(\mathcal{W})$  such that  $\tau_\omega(p) > 0$ . There exists a positive contraction  $(p_n)_n$  in  $\mathcal{W}_\omega$  such that  $p = [(p_n)_n]$ . Let  $a \in \mathcal{W}_{+,1} \setminus \{0\}$ ,  $b \in (\mathcal{W}_p^\omega)_{+,1}$  and  $\varepsilon > 0$ . Since  $\mathcal{W}_p^\omega = \overline{\rho(p \otimes s)\mathcal{W}^\omega \rho(p \otimes s)} = \overline{(p_n s)_n \mathcal{W}^\omega (s p_n)_n}$  where  $s$  is a strictly positive element in  $\mathcal{W}$ , we have  $(p_n)_n b (p_n)_n = b$ . Hence  $d_{\tau_\omega}(b) \leq \tau_\omega(p)$ . By Lemma 3.6 and  $\tau_\omega(p) > 0$ , we have

$$\tau_\omega(\sigma_p(a)) = \tau_\omega(p)\tau(a) > \tau_\omega(p)\frac{\tau(a)}{2} \geq d_{\tau_\omega}(b)\frac{\tau(a)}{2}.$$

Since  $\mathcal{W}$  has strict comparison and the tracial state on  $\mathcal{W}$  is unique,  $\mathcal{W}^\omega$  has strict comparison with respect to  $\{\tau_\omega\}$  (see, for example, the proof of [2, Lemma 1.23]). Therefore Lemma 3.5 implies that there exist elements  $x_1, x_2, \dots, x_{L(a,\varepsilon)}$  in  $\mathcal{W}^\omega$  such that

$$\|x_i\| \leq N(a, \varepsilon)$$

for any  $i = 1, 2, \dots, L(a, \varepsilon)$  and

$$\|b - \sum_{i=1}^{L(a,\varepsilon)} x_i \sigma_p(a) x_i^*\| < \varepsilon.$$

Since  $\mathcal{W}_p^\omega$  is a hereditary subalgebra of  $\mathcal{W}^\omega$ , we obtain the conclusion by Lemma 3.4.  $\square$

The following corollary is an immediate consequence of Proposition 3.3 and Proposition 3.7.

**Corollary 3.8.** For any finite subsets  $F_1 \subset C(\mathbb{T})$ ,  $F_2 \subset \mathcal{W}$ ,  $\varepsilon > 0$ , there exist finite subsets  $G_1 \subset C(\mathbb{T})$ ,  $G_2 \subset \mathcal{W}$ ,  $m \in \mathbb{N}$  and  $\delta > 0$  such that the following hold. Let  $p$  be a projection in  $F(\mathcal{W})$  such that  $\tau_\omega(p) > 0$  where  $\tau$  is the unique tracial state on  $\mathcal{W}$ . For any contractive  $(G_1 \odot G_2, \delta)$ -multiplicative maps  $\varphi, \psi : C(\mathbb{T}) \otimes \mathcal{W} \rightarrow \mathcal{W}_p^\omega$ , there exist a unitary element  $u$  in  $M_{m^2+1}(\mathcal{W}_p^\omega)^\sim$  and  $z_1, z_2, \dots, z_m \in \mathbb{T}$  such that

$$\begin{aligned} & \left\| u^* \left( \varphi(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k) \rho(p \otimes a) \oplus \cdots \oplus \bigoplus_{k=1}^m f(z_k) \rho(p \otimes a)}^m \right) u \right. \\ & \quad \left. - \psi(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k) \rho(p \otimes a) \oplus \cdots \oplus \bigoplus_{k=1}^m f(z_k) \rho(p \otimes a)}^m \right\| < \varepsilon \end{aligned}$$

for any  $f \in F_1$  and  $a \in F_2$ .

#### 4. PROPERTIES OF $F(\mathcal{W})$

In this section we shall consider properties of  $F(\mathcal{W})$ . In the rest of this paper, we denote by  $\tau$  the unique tracial state on  $\mathcal{W}$ . Since  $\mathcal{W}$  has property (SI), the following proposition is an immediate consequence of Theorem 2.6.

**Proposition 4.1.** (i) The central sequence  $C^*$ -algebra  $F(\mathcal{W})$  has a unique tracial state  $\tau_\omega$ .

(ii) If  $a$  and  $b$  are positive elements in  $F(\mathcal{W})$  satisfying  $d_{\tau_\omega}(a) < d_{\tau_\omega}(b)$ , then there exists an element  $r \in F(\mathcal{W})$  such that  $r^*br = a$ . Moreover,  $F(\mathcal{W})$  has strict comparison.

The following proposition shows that  $F(\mathcal{W})$  has many projections.

**Proposition 4.2.** (i) For any  $N \in \mathbb{N}$ , there exists a unital homomorphism from  $M_N(\mathbb{C})$  to  $F(\mathcal{W})$ .

(ii) For any  $\theta \in [0, 1]$ , there exists a projection  $p$  in  $F(\mathcal{W})$  such that  $\tau_\omega(p) = \theta$ .

(iii) Let  $h$  be a positive element in  $F(\mathcal{W})$  such that  $d_{\tau_\omega}(h) > 0$ . For any  $\theta \in [0, d_{\tau_\omega}(h))$ , there exists a projection  $p$  in  $hF(\mathcal{W})h$  such that  $\tau_\omega(p) = \theta$ .

*Proof.* Let  $\{h_n\}_{n \in \mathbb{N}}$  be an approximate unit for  $\mathcal{W}$ .

(i) By Theorem 2.7,  $\mathcal{W}$  is isomorphic to  $\mathcal{W} \otimes M_{N^\infty} = \mathcal{W} \otimes \bigotimes_{n \in \mathbb{N}} M_N(\mathbb{C})$ . Define a map  $\varphi$  from  $M_N(\mathbb{C})$  to  $F(\mathcal{W}) \cong F(\mathcal{W} \otimes \bigotimes_{n \in \mathbb{N}} M_N(\mathbb{C}))$  by

$$\varphi(x) := [(h_n \otimes \overbrace{1 \otimes \cdots \otimes 1}^n \otimes x \otimes 1 \otimes \cdots)_n]$$

for any  $x \in M_N(\mathbb{C})$ . Then  $\varphi$  is a unital homomorphism.

(ii) Since  $\mathbb{Z}[1/2]$  is dense in  $\mathbb{R}$ , there exists a sequence  $\{q_n\}_{n \in \mathbb{N}}$  of projections in  $M_{2^\infty}$  such that  $\lim_{n \rightarrow \infty} \tau'(q_n) = \theta$ , where  $\tau'$  is the unique tracial state on  $M_{2^\infty}$ . Put

$$p := [(h_n \otimes \overbrace{1 \otimes \cdots \otimes 1}^n \otimes q_n \otimes 1 \otimes \cdots)_n] \in F(\mathcal{W} \otimes \bigotimes_{n \in \mathbb{N}} M_{2^\infty}) \cong F(\mathcal{W}).$$

Then  $p$  is a projection in  $F(\mathcal{W})$  such that  $\tau_\omega(p) = \theta$ .

(iii) There exists a projection  $q$  in  $F(\mathcal{W})$  such that  $\tau_\omega(q) = \theta$  by (ii). Proposition 4.1 implies that there exists an element  $r$  in  $F(\mathcal{W})$  such that  $rhr^* = q$ . Let  $p := h^{\frac{1}{2}}r^*rh^{\frac{1}{2}}$ , then  $p$  is a projection in  $hF(\mathcal{W})h$  such that  $\tau_\omega(p) = \theta$ .  $\square$

Recall that  $\ker(\varrho_{\mathcal{W}}) = \{x \in F(\mathcal{W}) \mid \tau_\omega(x^*x) = 0\}$ .

**Proposition 4.3.** Let  $x$  be an element in  $F(\mathcal{W})$ . Then  $x$  is full if and only if  $x \notin \ker(\varrho_{\mathcal{W}})$ .

*Proof.* It is obvious that if  $x \in \ker(\varrho_{\mathcal{W}})$ , then  $x$  is not full in  $F(\mathcal{W})$ . Let  $x \notin \ker(\varrho_{\mathcal{W}})$ , then  $\tau_{\omega}(x^*x) > 0$ . Using Lemma 3.5, it is easy to see that  $x$  is full because  $F(\mathcal{W})$  has strict comparison.  $\square$

Using the ideas in [47] and [46], we shall show that certain elements in  $F(\mathcal{W})$  can be approximated by invertible elements. We denote by  $\text{GL}(A)$  the set of invertible elements in  $A$ .

**Lemma 4.4.** Let  $a$  and  $b$  be positive elements in  $F(\mathcal{W})$  such that  $a, b \notin \ker(\varrho_{\mathcal{W}})$ . Then there exist a unitary element  $u$  and a projection  $p'$  in  $F(\mathcal{W})$  such that  $p' \notin \ker(\varrho_{\mathcal{W}})$  and  $p' \in \overline{aF(\mathcal{W})a} \cap u\overline{bF(\mathcal{W})b}u^*$ .

*Proof.* Since  $a, b \notin \ker(\varrho_{\mathcal{W}})$ , we have  $d_{\tau_{\omega}}(a) > 0$  and  $d_{\tau_{\omega}}(b) > 0$ . Proposition 4.2 and Proposition 4.1 imply that there exist a projection  $p \in \overline{aF(\mathcal{W})a}$  and an element  $r \in F(\mathcal{W})$  such that  $p \notin \ker(\varrho_{\mathcal{W}})$  and  $rbr^* = p$ . Then we have  $prb \notin \ker(\varrho_{\mathcal{W}})$ , and hence  $arb \notin \ker(\varrho_{\mathcal{W}})$ . Since  $r$  is a linear combination of four unitaries in  $F(\mathcal{W})$ , there exists a unitary element  $w$  in  $F(\mathcal{W})$  such that  $awb \notin \ker(\varrho_{\mathcal{W}})$ . Since  $\ker(\varrho_{\mathcal{W}})$  is closed, essentially the same argument as in the proof of [47, Lemma 3.4] show that there exist a unitary element  $u$  and a positive element  $c \in F(\mathcal{W})$  such that  $c \notin \ker(\varrho_{\mathcal{W}})$  and  $c \in \overline{aF(\mathcal{W})a} \cap u\overline{bF(\mathcal{W})b}u^*$ . Using Proposition 4.2, we can find a projection  $p'$  satisfying the conclusion of the proposition.  $\square$

**Lemma 4.5.** Let  $x$  be an element in  $F(\mathcal{W})$ . Assume that there exist projections  $p$  and  $q$  in  $F(\mathcal{W})$  such that  $xp = qx = x$  and  $1 - p, 1 - q \notin \ker(\varrho_{\mathcal{W}})$ . Then there exist a unitary element  $u$  and a projection  $e$  in  $F(\mathcal{W})$  such that  $eux = uxe = ux$  and  $1 - e \notin \ker(\varrho_{\mathcal{W}})$ .

*Proof.* Lemma 4.4 implies that there exist a unitary element  $u$  and a projection  $p'$  in  $F(\mathcal{W})$  such that  $p' \notin \ker(\varrho_{\mathcal{W}})$  and

$$p' \in (1 - p)F(\mathcal{W})(1 - p) \cap u(1 - q)F(\mathcal{W})(1 - q)u^*.$$

Since  $x(1 - p) = (1 - q)x = 0$ , we have  $uxp' = 0$  and  $p'ux = 0$ . Put  $e := 1 - p'$ , then we obtain the conclusion.  $\square$

**Lemma 4.6.** Let  $y$  be an element in  $F(\mathcal{W})$ . Assume that there exists a projection  $e$  in  $F(\mathcal{W})$  such that  $ey = ye = y$  and  $1 - e \notin \ker(\varrho_{\mathcal{W}})$ . Then  $y \in \overline{\text{GL}(F(\mathcal{W}))}$ .

*Proof.* It is enough to show that  $y$  is a product of two nilpotent elements. Since  $1 - e$  is full by Proposition 4.3, there exist elements  $x_1, x_2, \dots, x_N$  in  $F(\mathcal{W})$  such that  $\sum_{i=1}^N x_i(1 - e)x_i^* = 1$ . Proposition 4.2 implies that there exists a unital homomorphism  $\varphi$  from  $M_{N+1}(\mathbb{C})$  to  $F(\mathcal{W})$ . Let  $\{e_{ij}\}_{i,j=1}^{N+1}$  be the standard matrix units of  $M_{N+1}(\mathbb{C})$ . Taking suitable subsequences of representatives of  $\varphi(e_{ij})$  for any  $i, j = 1, 2, \dots, N+1$ , we may assume that the range of  $\varphi$  commutes with  $y$ . Put

$$r := \sum_{i=1}^N ex_i(1 - e)\varphi(e_{N+1i}) + \sum_{i=1}^N y\varphi(e_{ii+1})$$

and

$$t := \sum_{i=1}^N (1 - e)x_i^*y\varphi(e_{iN+1}) + \sum_{i=1}^N e\varphi(e_{i+1i}).$$

Then we see that  $rt = y$  and  $r^{N+2} = t^{N+2} = 0$ .  $\square$

The following proposition is an immediate consequence of Lemma 4.5 and Lemma 4.6.

**Proposition 4.7.** Let  $x$  be an element in  $F(\mathcal{W})$ . Assume that there exist projections  $p$  and  $q$  in  $F(\mathcal{W})$  such that  $xp = qx = x$  and  $1 - p, 1 - q \notin \ker(\varrho_{\mathcal{W}})$ . Then  $x \in \overline{\text{GL}(F(\mathcal{W}))}$ .

Using the proposition above, we shall show that for certain projections in  $F(\mathcal{W})$ , Murray-von Neumann equivalence and unitary equivalence coincide.

**Proposition 4.8.** Let  $p$  and  $q$  be projections in  $F(\mathcal{W})$  such that  $\tau_{\omega}(p) < 1$ . Then  $p$  and  $q$  are Murray-von Neumann equivalent if and only if  $p$  and  $q$  are unitarily equivalent.

*Proof.* The if part is obvious. We will show the only if part. Suppose that there exists a partial isometry  $v \in F(\mathcal{W})$  such that  $v^*v = p$  and  $vv^* = q$ . Since we have  $vp = qv = v$  and  $\tau_{\omega}(1 - q) = \tau_{\omega}(1 - p) > 0$ , there exists an invertible element  $s$  of norm one such that  $\|s - v\| < 1/4$  by Proposition 4.7. Let  $u := s(s^*s)^{-\frac{1}{2}}$ . Then  $u$  is a unitary element in  $F(\mathcal{W})$  and we have  $\|upu^* - q\| < 1$ . Therefore we see that  $p$  is unitarily equivalent to  $q$ .  $\square$

We shall show that every unitary element in  $F(\mathcal{W})$  can be lifted to a unitary element in  $\mathcal{W}_{\omega}^{\sim}$ .

**Proposition 4.9.** Let  $A$  be a  $\sigma$ -unital  $C^*$ -algebra with  $A \subset \overline{\text{GL}(A^{\sim})}$ . If  $u$  is a unitary element in  $F(A)$ , then there exists a unitary element  $w$  in  $A_{\omega}^{\sim}$  such that  $u = [w]$ .

*Proof.* Since  $A \subset \overline{\text{GL}(A^{\sim})}$ , there exists a bounded sequence  $\{z_n\}_{n \in \mathbb{N}}$  of invertible elements in  $A^{\sim}$  such that  $u = [(z_n)_n]$ . Note that for any  $a \in A$ ,

$$\lim_{n \rightarrow \omega} \|z_n^* z_n a - a\| = 0, \quad \lim_{n \rightarrow \omega} \|z_n z_n^* a - a\| = 0$$

because  $u$  is a unitary element in  $F(A)$ . For any  $n \in \mathbb{N}$ , let  $w_n := z_n(z_n^* z_n)^{-\frac{1}{2}}$ . Then  $w_n$  is a unitary element in  $A^{\sim}$  and for any  $a \in A$ , we have

$$\|w_n a - z_n a\| = \|w_n(a - (z_n^* z_n)^{\frac{1}{2}} a)\| = \|a - (z_n^* z_n)^{\frac{1}{2}} a\| \rightarrow 0$$

as  $n \rightarrow \omega$ . Furthermore, for any  $a \in A$ , we have

$$\begin{aligned} \|[w_n, a]\| &= \|w_n a w_n^* - a\| \\ &= \|w_n a w_n^* - z_n a w_n^* + z_n a w_n^* - z_n a z_n^* + z_n a z_n^* - z_n z_n^* a + z_n z_n^* a - a\| \\ &\leq \|w_n a - z_n a\| + \|z_n\| \|w_n a^* - z_n a^*\| + \|z_n\| \|a z_n^* - z_n^* a\| + \|z_n z_n^* a - a\| \\ &\rightarrow 0 \end{aligned}$$

as  $n \rightarrow \omega$ . Therefore  $(w_n)_n$  is a unitary element in  $A_{\omega}^{\sim}$  such that  $u = [(w_n)_n]$ .  $\square$

Since  $\mathcal{W}$  has stable rank one, we have the following corollary.

**Corollary 4.10.** Let  $u$  be a unitary element in  $F(\mathcal{W})$ . Then there exists a unitary element  $w$  in  $\mathcal{W}_{\omega}^{\sim}$  such that  $u = [w]$ .

## 5. HOMOTOPY OF UNITARIES IN $F(\mathcal{W})$

In this section we shall prove Theorem 5.7. The following lemma is motivated by [35, Lemma 4.1] and [35, Lemma 4.2].

**Lemma 5.1.** Let  $F$  be a finite subset of  $C(\mathbb{T})$  and  $\varepsilon > 0$ . Suppose that  $\varphi$  and  $\psi$  are unital homomorphisms from  $C(\mathbb{T})$  to  $F(\mathcal{W})$  such that  $\tau_{\omega} \circ \varphi = \tau_{\omega} \circ \psi$ . Then there exist a projection  $p \in F(\mathcal{W})$ ,  $(F, \varepsilon)$ -multiplicative unital c.p. maps  $\varphi'$  and  $\psi'$  from  $C(\mathbb{T})$  to  $pF(\mathcal{W})p$  and a unital homomorphism  $\sigma$  from  $C(\mathbb{T})$  to  $(1 - p)F(\mathcal{W})(1 - p)$  with finite-dimensional range such that

$$0 < \tau_{\omega}(p) < \varepsilon, \quad \varphi \sim_{F, \varepsilon} \varphi' \oplus \sigma, \quad \psi \sim_{F, \varepsilon} \psi' \oplus \sigma.$$

*Proof.* We may assume that every element in  $F$  is of norm one. Let  $\mu$  be the probability measure on  $\mathbb{T}$  corresponding to  $\tau_\omega \circ \varphi = \tau_\omega \circ \psi$ . By the same argument as in the proof of [35, Lemma 4.1], there exist pairwise disjoint open subsets  $W_1, W_2, \dots, W_l \subset \mathbb{T}$  such that

$$\mu(\mathbb{T} \setminus \bigcup_{i=1}^l W_i) = 0, \quad \mu(W_i) > 0$$

and  $|f(x) - f(y)| < \varepsilon/3$  for any  $x, y \in W_i$  and  $f \in F$ . For any  $i = 1, 2, \dots, l$ , choose  $z_i \in W_i$ . Proposition 4.2 implies that there exists a projection  $p_i$  in  $\overline{\varphi(C_0(W_i))F(\mathcal{W})\varphi(C_0(W_i))}$  such that

$$\mu(W_i) - \frac{\varepsilon}{l} < \tau_\omega(p_i) < \mu(W_i).$$

Note that we have

$$\|p_i \varphi(f) - f(z_i) p_i\| < \frac{\varepsilon}{3}, \quad \|\varphi(f) p_i - f(z_i) p_i\| < \frac{\varepsilon}{3}$$

for any  $f \in F$  and  $i = 1, 2, \dots, l$ . In the same way as in the proof of [35, Lemma 4.1], we see that there exist mutually orthogonal projections  $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_l$  in  $F(\mathcal{W})^{**}$  such that  $\bar{p}_i$  commutes with  $\varphi(C(\mathbb{T}))$  and  $p_i \leq \bar{p}_i$ . In a similar way as above, there exist a projection  $q_i$  in  $\overline{\psi(C_0(W_i))F(\mathcal{W})\psi(C_0(W_i))}$  such that

$$\tau_\omega(p_i) < \tau_\omega(q_i) < \mu(W_i)$$

and

$$\|q_i \psi(f) - f(z_i) q_i\| < \frac{\varepsilon}{3}, \quad \|\psi(f) q_i - f(z_i) q_i\| < \frac{\varepsilon}{3}$$

for any  $f \in F$  and  $i = 1, 2, \dots, l$ . Also, there exist mutually orthogonal projections  $\bar{q}_1, \bar{q}_2, \dots, \bar{q}_l$  in  $F(\mathcal{W})^{**}$  such that  $\bar{q}_i$  commutes with  $\psi(C(\mathbb{T}))$  and  $q_i \leq \bar{q}_i$ .

For any  $i = 1, 2, \dots, l$ , there exists a subprojection  $q'_i$  of  $q_i$  such that  $q'_i$  is Murray-von Neumann equivalent to  $p_i$  by Proposition 4.1. It follows from Proposition 4.8 that there exists a unitary element  $u$  in  $F(\mathcal{W})$  such that  $u p_i u^* = q'_i$  for any  $i = 1, 2, \dots, l$ . Put  $p := 1 - \sum_{i=1}^l p_i$  and  $q := 1 - \sum_{i=1}^l q'_i$ . Then we have

$$0 < \tau_\omega(p) < \varepsilon.$$

Since  $(1-p)\varphi(f) = \sum_{i=1}^l p_i \varphi(f) = \sum_{i=1}^l p_i \bar{p}_i \varphi(f) = \sum_{i=1}^l p_i \varphi(f) \bar{p}_i$ , we have

$$\|(1-p)\varphi(f) - \sum_{i=1}^l f(z_i) p_i\| < \frac{\varepsilon}{3}$$

for any  $f \in F$ . Moreover, we have

$$\|[p, \varphi(f)]\| < \frac{2\varepsilon}{3}$$

for any  $f \in F$ . In a similar way, we have

$$\|(1-q)\psi(f) - \sum_{i=1}^l f(z_i) q'_i\| < \frac{\varepsilon}{3}, \quad \|[q, \psi(f)]\| < \frac{2\varepsilon}{3}$$

for any  $f \in F$ .

Define unital c.p. maps  $\varphi'$  and  $\psi'$  from  $C(\mathbb{T})$  to  $pF(\mathcal{W})p$  by

$$\varphi'(f) := p\varphi(f)p, \quad \psi'(f) := pu\psi(f)u^*p,$$

and define a unital homomorphism  $\sigma$  from  $C(\mathbb{T})$  to  $(1-p)F(\mathcal{W})(1-p)$  by

$$\sigma(f) := \sum_{i=1}^l f(z_i) p_i.$$

Then it is easy to see that  $\varphi'$  and  $\psi'$  are  $(F, 2\varepsilon/3)$ -multiplicative maps. We have

$$\begin{aligned} \|\varphi(f) - (\varphi'(f) + \sigma(f))\| &\leq \|p\varphi(f) - p\varphi'(f)p\| + \|(1-p)\varphi(f) - \sum_{i=1}^l f(z_i)p_i\| \\ &< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for any  $f \in F$ . Also, we have

$$\|\psi(f) - u^*(\psi'(f) + \sigma(f))u\| = \|\psi(f) - q\psi(f)q - \sum_{i=1}^l f(z_i)q'_i\| < \varepsilon$$

for any  $f \in F$ . Therefore the proof is complete.  $\square$

The following theorem is related to [35, Theorem 4.5].

**Theorem 5.2.** Let  $F_1$  be a finite subset of  $C(\mathbb{T})$  and  $F_2$  a finite subset of  $\mathcal{W}$ , and let  $\varepsilon > 0$ . Then there exist mutually orthogonal positive elements  $h_1, h_2, \dots, h_l$  in  $C(\mathbb{T})$  of norm one such that the following holds. For any  $\nu > 0$ , there exist finite subsets  $G_1 \subset C(\mathbb{T})$ ,  $G_2 \subset \mathcal{W}$  and  $\delta > 0$  such that the following holds. If  $\varphi$  and  $\psi$  are unital homomorphisms from  $C(\mathbb{T})$  to  $\mathcal{W}^\sim$  such that

$$\begin{aligned} \tau(\varphi(h_i)) &\geq \nu, \quad 1 \leq i \leq l, \\ \|[\varphi(f), a]\| &< \delta, \quad \|[\psi(f), a]\| < \delta, \quad \forall f \in G_1, a \in G_2, \\ |\tau(\varphi(f)) - \tau(\psi(f))| &< \delta, \quad \forall f \in G_1, \end{aligned}$$

then there exists a unitary element  $u$  in  $\mathcal{W}^\sim$  such that

$$\|u\varphi(f)au^* - \psi(f)a\| < \varepsilon$$

for any  $f \in F_1$  and  $a \in F_2$ .

*Proof.* We may assume that every element in  $F_2$  is of norm one. Let  $\{y_1, y_2, \dots, y_l\}$  be a finite subset of  $\mathbb{T}$  such that for any  $x \in \mathbb{T}$ , there exists  $y_i \in \{y_1, y_2, \dots, y_l\}$  such that  $|f(x) - f(y_i)| < \varepsilon/7$  for any  $f \in F_1$ . Choose pairwise disjoint open neighborhoods  $W_1, W_2, \dots, W_l$  of  $y_1, y_2, \dots, y_l$  respectively such that if  $x \in W_i$  and  $f \in F_1$ , then  $|f(x) - f(y_i)| < \varepsilon/7$ . For any  $i = 1, 2, \dots, l$ , take a positive element  $h_i \in C_0(W_i)$  of norm one. We shall show that  $h_1, h_2, \dots, h_l$  have the desired property. On the contrary, suppose that  $h_1, h_2, \dots, h_l$  did not have the desired property. Then there exists a positive number  $\nu$  satisfying the following: For any  $n \in \mathbb{N}$ , there exist unital homomorphisms  $\varphi_n, \psi_n : C(\mathbb{T}) \rightarrow \mathcal{W}^\sim$  such that

$$\tau(\varphi_n(h_i)) \geq \nu, \quad 1 \leq i \leq l,$$

$$\|[\varphi_n(f), a]\| \rightarrow 0, \quad \|[\psi_n(f), a]\| \rightarrow 0, \quad |\tau(\varphi_n(f)) - \tau(\psi_n(f))| \rightarrow 0$$

as  $n \rightarrow \infty$  for any  $f \in C(\mathbb{T})$ ,  $a \in \mathcal{W}$  and

$$\max_{f \in F_1, a \in F_2} \|u\varphi_n(f)au^* - \psi_n(f)a\| \geq \varepsilon$$

for any unitary element  $u$  in  $\mathcal{W}^\sim$ .

Define homomorphisms  $\varphi$  and  $\psi$  from  $C(\mathbb{T})$  to  $F(\mathcal{W})$  by

$$\varphi(f) := [(\varphi_n(f))_n], \quad \psi(f) := [(\psi_n(f))_n]$$

for any  $f \in C(\mathbb{T})$ , and define homomorphisms  $\Phi$  and  $\Psi$  from  $C(\mathbb{T}) \otimes \mathcal{W}$  to  $\mathcal{W}^\omega$  by

$$\Phi := \rho \circ (\varphi \otimes \text{id}_{\mathcal{W}}), \quad \Psi := \rho \circ (\psi \otimes \text{id}_{\mathcal{W}}).$$

Note that we have

$$\tau_\omega(\varphi(h_i)) \geq \nu$$

for any  $i = 1, 2, \dots, l$  and

$$\tau_\omega \circ \varphi = \tau_\omega \circ \psi.$$



Applying Corollary 3.8 to  $F_1$ ,  $F_2$  and  $\varepsilon/7$ , we obtain finite subsets  $G_1 \subset C(\mathbb{T})$ ,  $G_2 \subset \mathcal{W}$ ,  $m \in \mathbb{N}$  and  $\delta > 0$ . Put

$$F'_1 := F_1 \cup G_1 \cup \{h_1, h_2, \dots, h_l\}, \quad \varepsilon' := \min\{\varepsilon/7, \delta, \nu/(m^2 + 2)\}.$$

Applying Lemma 5.1 to  $F'_1$ ,  $\varepsilon'$ ,  $\varphi$  and  $\psi$ , there exist a projection  $p \in F(\mathcal{W})$ ,  $(F', \varepsilon')$ -multiplicative unital c.p. maps  $\varphi'$  and  $\psi'$  from  $C(\mathbb{T})$  to  $pF(\mathcal{W})p$  and a unital homomorphism  $\sigma$  from  $C(\mathbb{T})$  to  $(1-p)F(\mathcal{W})(1-p)$  with finite-dimensional range such that

$$0 < \tau_\omega(p) < \varepsilon', \quad \varphi \sim_{F', \varepsilon'} \varphi' \oplus \sigma, \quad \psi \sim_{F', \varepsilon'} \psi' \oplus \sigma.$$

Define c.p. maps  $\Phi'$  and  $\Psi'$  from  $C(\mathbb{T}) \otimes \mathcal{W}$  to  $\mathcal{W}_p^\omega$  by

$$\Phi' := \rho \circ (\varphi' \otimes \text{id}_\mathcal{W}), \quad \Psi' := \rho \circ (\psi' \otimes \text{id}_\mathcal{W})$$

and define a homomorphism  $\Sigma$  from  $C(\mathbb{T}) \otimes \mathcal{W}$  to  $\mathcal{W}_{1-p}^\omega$  by

$$\Sigma := \rho \circ (\sigma \otimes \text{id}_\mathcal{W}).$$

Using Corollary 4.10, we see that

$$\Phi \sim_{F_1 \odot F_2, \frac{\varepsilon}{7}} \Phi' \oplus \Sigma, \quad \Psi \sim_{F_1 \odot F_2, \frac{\varepsilon}{7}} \Psi' \oplus \Sigma.$$

It can be easily checked that  $\Phi'$  and  $\Psi'$  are contractive  $(G_1 \odot G_2, \delta)$ -multiplicative maps. By Corollary 3.8, there exist a unitary element  $U$  in  $M_{m^2+1}(\mathcal{W}_p^\omega)^\sim$  and  $z_1, z_2, \dots, z_m \in \mathbb{T}$  such that

$$\begin{aligned} & \left\| U^* \left( \Phi'(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k) \rho(p \otimes a) \oplus \dots \oplus \bigoplus_{k=1}^m f(z_k) \rho(p \otimes a)}^m \right) U \right. \\ & \quad \left. - \Psi'(f \otimes a) \oplus \overbrace{\bigoplus_{k=1}^m f(z_k) \rho(p \otimes a) \oplus \dots \oplus \bigoplus_{k=1}^m f(z_k) \rho(p \otimes a)}^m \right\| < \frac{\varepsilon}{7} \end{aligned}$$

for any  $f \in F_1$  and  $a \in F_2$ .

For any homomorphism  $\gamma : C(\mathbb{T}) \rightarrow F(\mathcal{W})$ , let  $\mu_\gamma$  denote the probability measure on  $\mathbb{T}$  corresponding to  $\tau_\omega \circ \gamma$ . For any  $i = 1, 2, \dots, l$ , we have

$$\mu_\sigma(W_i) \geq \tau_\omega(\sigma(h_i)) > \tau_\omega(\varphi(h_i)) - \tau_\omega(\varphi'(h_i)) - \varepsilon' \geq \nu - \tau_\omega(p) - \varepsilon' > \nu - 2\varepsilon' \geq m^2\varepsilon'.$$

Hence we see that there exists a homomorphism  $\sigma' : C(\mathbb{T}) \rightarrow (1-p)F(\mathcal{W})(1-p)$  with finite-dimensional range such that

$$\|\sigma(f) - \sigma'(f)\| < \frac{\varepsilon}{7}$$

for any  $f \in F_1$  and  $\mu_{\sigma'}(\{y_i\}) > m^2\varepsilon'$  for any  $i = 1, 2, \dots, l$  because of the property of  $W_i$ . Using Proposition 4.1, we see that there exist mutually orthogonal projections  $\{p_{j,k}\}_{j,k=1}^m$  in  $(1-p)F(\mathcal{W})(1-p)$  and a homomorphism  $\sigma'' : C(\mathbb{T}) \rightarrow (1-p-q)F(\mathcal{W})(1-p-q)$  where  $q = \sum_{j,k=1}^m p_{j,k}$  such that

$$\|\sigma'(f) - (\sum_{j=1}^m \sum_{k=1}^m f(z_k) p_{j,k} + \sigma''(f))\| < \frac{\varepsilon}{7}$$

for any  $f \in F_1$  and  $p_{j,k}$  is Murray-von Neumann equivalent to  $p$  for any  $j, k = 1, 2, \dots, m$  because of the property of  $\{y_1, y_2, \dots, y_l\}$ . By the argument above, it can

be checked that there exists a unitary element  $\widehat{U}$  in  $(\mathcal{W}_{p+q}^\omega)^\sim$  such that

$$\begin{aligned} & \|\widehat{U}^*(\Phi'(f \otimes a) + \sum_{j=1}^m \sum_{k=1}^m f(z_k) \rho(p_{j,k} \otimes a)) \widehat{U} \\ & \quad - \Psi'(f \otimes a) + \sum_{j=1}^m \sum_{k=1}^m f(z_k) \rho(p_{j,k} \otimes a)\| < \frac{\varepsilon}{7}. \end{aligned}$$

Note that we may assume that  $\widehat{U} = (a_n)_n + 1$  for some  $(a_n)_n \in \mathcal{W}_{p+q}^\omega$ . Therefore we see that there exists a unitary element  $V$  in  $(\mathcal{W}^\omega)^\sim$  such that

$$\begin{aligned} & \|V^*(\Phi'(f \otimes a) + \sum_{j=1}^m \sum_{k=1}^m f(z_k) \rho(p_{j,k} \otimes a) + \sigma''(f))V \\ & \quad - (\Psi'(f \otimes a) + \sum_{j=1}^m \sum_{k=1}^m f(z_k) \rho(p_{j,k} \otimes a) + \sigma''(f))\| < \frac{\varepsilon}{7}. \end{aligned}$$

Let  $\Sigma'$  and  $\Sigma''$  be homomorphisms from  $C(\mathbb{T}) \otimes \mathcal{W}$  to  $\mathcal{W}_{1-p}^\omega$  such that

$$\Sigma'(f \otimes a) = \rho(\sigma'(f) \otimes a), \quad \Sigma''(f \otimes a) = \rho\left(\sum_{j=1}^m \sum_{k=1}^m (f(z_k) p_{j,k} + \sigma''(f)) \otimes a\right)$$

for any  $f \in C(\mathbb{T})$  and  $a \in \mathcal{W}$ . Then we have

$$\begin{aligned} \Phi & \sim_{F_1 \odot F_2, \frac{\varepsilon}{7}} \Phi' \oplus \Sigma \sim_{F_1 \odot F_2, \frac{\varepsilon}{7}} \Phi' \oplus \Sigma' \sim_{F_1 \odot F_2, \frac{\varepsilon}{7}} \Phi' \oplus \Sigma'' \sim_{F_1 \odot F_2, \frac{\varepsilon}{7}} \Psi' \oplus \Sigma'' \\ & \sim_{F_1 \odot F_2, \frac{\varepsilon}{7}} \Psi' \oplus \Sigma' \sim_{F_1 \odot F_2, \frac{\varepsilon}{7}} \Psi' \oplus \Sigma \sim_{F_1 \odot F_2, \frac{\varepsilon}{7}} \Psi. \end{aligned}$$

Therefore there exists a unitary element  $(w_n)_n$  in  $(\mathcal{W}^\omega)^\sim$  such that

$$\|(w_n)_n \Phi(f \otimes a) (w_n)_n^* - \Psi(f \otimes a)\| < \varepsilon$$

for any  $f \in F_1$  and  $a \in F_2$ . Note that we may assume that  $w_n$  is a unitary element in  $\mathcal{W}^\sim$  for any  $n \in \mathbb{N}$ . Taking a sufficiently large  $n$ , we obtain a contradiction. Consequently, the proof is complete.  $\square$

We denote by  $\iota$  the identity function on  $\mathbb{T}$ , that is,  $\iota(z) = z$  for any  $z \in \mathbb{T}$ .

**Theorem 5.3.** Let  $u$  and  $v$  be unitaries in  $F(\mathcal{W})$  such that  $\tau_\omega(f(u)) > 0$  for any  $f \in C(\mathbb{T})_+ \setminus \{0\}$ . Then there exists a unitary element  $w$  in  $F(\mathcal{W})$  such that  $wuw^* = v$  if and only if  $\tau_\omega(f(u)) = \tau_\omega(f(v))$  for any  $f \in C(\mathbb{T})$ .

*Proof.* The only if part is obvious. We will show the if part. By Corollary 4.10, there exist unitaries  $(u_n)_n$  and  $(v_n)_n$  in  $\mathcal{W}_\omega^\sim$  such that  $u = [(u_n)_n]$  and  $v = [(v_n)_n]$ . For any  $n \in \mathbb{N}$ , define unital homomorphisms  $\varphi_n$  and  $\psi_n$  from  $C(\mathbb{T})$  to  $\mathcal{W}^\sim$  by  $\varphi_n(f) := f(u_n)$  and  $\psi_n(f) := f(v_n)$ , respectively. Then we have

$$\begin{aligned} |\tau(\varphi_n(f)) - \tau_\omega(f(u))| & \rightarrow 0, \quad \|[\varphi_n(f), a]\| \rightarrow 0, \quad \|[\psi_n(f), a]\| \rightarrow 0, \\ |\tau(\varphi_n(f)) - \tau(\psi_n(f))| & \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \omega$  for any  $f \in C(\mathbb{T})$  and  $a \in \mathcal{W}$ .

Let  $F_1 := \{1, \iota\} \subset C(\mathbb{T})$ , and let  $\{F_{2,k}\}_{k \in \mathbb{N}}$  be a sequence of finite subsets of  $\mathcal{W}$  such that  $F_{2,k} \subset F_{2,k+1}$  and  $\mathcal{W} = \overline{\bigcup_{k \in \mathbb{N}} F_{2,k}}$ . For any  $k \in \mathbb{N}$ , applying Theorem 5.2 to  $F_1$ ,  $F_{2,k}$  and  $1/k$ , we obtain mutually orthogonal positive elements  $h_{1,k}, h_{2,k}, \dots, h_{l(k),k}$  in  $C(\mathbb{T})$  of norm one. Let

$$\nu_k := \frac{1}{2} \min\{\tau_\omega(h_{1,k}(u)), \tau_\omega(h_{2,k}(u)), \dots, \tau_\omega(h_{l(k),k}(u))\} > 0.$$

Applying Theorem 5.2 to  $\nu_k$ , we obtain finite subsets  $G_{1,k} \subset C(\mathbb{T})$ ,  $G_{2,k} \subset \mathcal{W}$  and  $\delta_k > 0$ . We may assume that  $G_{1,k} \subset G_{1,k+1}$ ,  $G_{2,k} \subset G_{2,k+1}$  and  $\delta_k > \delta_{k+1}$ . It can

be easily checked that there exists a sequence  $\{X_k\}_{k \in \mathbb{N}}$  of elements in  $\omega$  such that  $X_k \supset X_{k+1}$  and for any  $n \in X_k$ ,

$$\begin{aligned} |\tau(\varphi_n(h_{i,k})) - \tau_\omega(h_{i,k}(u))| &< \frac{\tau_\omega(h_{i,k}(u))}{2}, \quad 1 \leq \forall i \leq l(k), \\ \|\varphi_n(f), a\| &< \delta_k, \quad \|\psi_n(f), a\| < \delta_k, \quad \forall f \in G_{1,k}, a \in G_{2,k}, \\ |\tau(\varphi_n(f)) - \tau(\psi_n(f))| &< \delta_k, \quad \forall f \in G_{1,k}. \end{aligned}$$

Since we have  $\tau(\varphi_n(h_{i,k})) \geq \nu_k$  by the above, Theorem 5.2 implies that for any  $n \in X_k$ , there exists a unitary element  $w_{k,n}$  in  $\mathcal{W}^\sim$  such that

$$\|w_{k,n}\varphi_n(f)aw_{k,n}^* - \psi_n(f)a\| < \frac{1}{k}$$

for any  $f \in F_1$  and  $a \in F_{2,k}$ . Since  $F_1 = \{1, \iota\}$ , we have

$$\|[w_{k,n}, a]\| < \frac{1}{k}, \quad \|w_{k,n}u_naw_{k,n}^* - v_na\| < \frac{1}{k}$$

for any  $n \in X_k$  and  $a \in F_{2,k}$ . Let

$$w_n := \begin{cases} 1 & \text{if } n \notin X_1 \\ w_{k,n} & \text{if } n \in X_k \setminus X_{k+1} \quad (k \in \mathbb{N}) \end{cases}.$$

Then we have

$$\|[w_n, a]\| \rightarrow 0, \quad \|w_nu_nw_n^*a - v_na\| \rightarrow 0$$

as  $n \rightarrow \omega$  for any  $a \in \mathcal{W}$ . Therefore  $[(w_n)_n]$  is a unitary element in  $F(\mathcal{W})$  and

$$[(w_n)_n]u[(w_n)_n]^* = v.$$

□

Hiroki Matui told us the following lemma.

**Lemma 5.4.** For any faithful tracial state  $\tau_0$  on  $C(\mathbb{T})$ , there exists a unital homomorphism  $\varphi$  from  $C(\mathbb{T})$  to  $M_{2^\infty}$  such that  $\tau_0 = \tau' \circ \varphi$  where  $\tau'$  is the unique tracial state on  $M_{2^\infty}$ .

*Proof.* We identify  $C(\mathbb{T})$  with  $\{f \in C([0, 1]) \mid f(0) = f(1)\}$ . Note that  $\tau_0$  extends to a faithful tracial state  $\tilde{\tau}_0$  on  $C([0, 1])$ . By [48, Theorem 2.1 (i)], there exists a unital homomorphism  $\psi$  from  $C([0, 1])$  to  $\mathcal{Z}$  such that  $\tilde{\tau}_0 = \tau_{\mathcal{Z}} \circ \psi$ , where  $\tau_{\mathcal{Z}}$  is the unique tracial state on  $\mathcal{Z}$ . Define a unital homomorphism  $\varphi$  from  $C(\mathbb{T})$  to  $M_{2^\infty} \otimes \mathcal{Z}$  by  $\varphi := 1 \otimes \psi|_{C(\mathbb{T})}$ . Since  $M_{2^\infty}$  is  $\mathcal{Z}$ -stable, we obtain the conclusion. □

Note that we identify  $F(\mathcal{W})$  with  $F(\mathcal{W} \otimes M_{2^\infty})$  in the following lemmas.

**Lemma 5.5.** Let  $u$  be a unitary element in  $F(\mathcal{W})$  such that  $\tau_\omega(f(u)) > 0$  for any  $f \in C(\mathbb{T})_+ \setminus \{0\}$ . Then there exist a unitary element  $(v_n)_n$  in  $(M_{2^\infty})_\omega$  and a unitary element  $w$  in  $F(\mathcal{W})$  such that

$$wuw^* = [(h_n \otimes v_n)_n]$$

where  $\{h_n\}_{n \in \mathbb{N}}$  is an approximate unit for  $\mathcal{W}$ .

*Proof.* By Lemma 5.4, there exists a unital homomorphism  $\varphi$  from  $C(\mathbb{T})$  to  $M_{2^\infty}$  such that  $\tau'(\varphi(f)) = \tau_\omega(f(u))$  for any  $f \in C(\mathbb{T})$ , where  $\tau'$  is the unique tracial state on  $M_{2^\infty}$ . For any  $n \in \mathbb{N}$ , let

$$v_n := \overbrace{1 \otimes \cdots \otimes 1}^n \otimes \varphi(\iota) \otimes 1 \otimes \cdots \in \bigotimes_{n \in \mathbb{N}} M_{2^\infty} \cong M_{2^\infty}.$$

Then  $(v_n)_n$  is a unitary element in  $(M_{2^\infty})_\omega$  and we have

$$\tau_\omega(f(u)) = \tau_\omega(f([(h_n \otimes v_n)_n]))$$

for any  $f \in C(\mathbb{T})$ . Therefore we obtain the conclusion by Theorem 5.3. □

For a Lipschitz continuous map  $U$ , we denote by  $\text{Lip}(U)$  its Lipschitz constant.

**Lemma 5.6.** Let  $u$  be a unitary element in  $F(\mathcal{W})$  such that  $\tau_\omega(f(u)) > 0$  for any  $f \in C(\mathbb{T})_+ \setminus \{0\}$ , and let  $z_0 \in \mathbb{T}$ . Then there exists a path of unitaries  $U : [0, 1] \rightarrow F(\mathcal{W})$  such that

$$U(0) = z_0 1, \quad U(1) = u, \quad \text{Lip}(U) \leq \pi.$$

*Proof.* Let  $\{h_n\}_{n \in \mathbb{N}}$  be an approximate unit for  $A$ . By Lemma 5.5, there exist a unitary element  $(v_n)_n$  in  $(M_{2^\infty})_\omega$  and a unitary element  $w$  in  $F(\mathcal{W})$  such that  $wuw^* = [(h_n \otimes v_n)_n]$ . There exists a path of unitaries  $V : [0, 1] \rightarrow (M_{2^\infty})_\omega$  such that

$$V(0) = z_0 1, \quad V(1) = (v_n)_n, \quad \text{Lip}(V) \leq \pi.$$

(See, for example, [16, Lemma 1].) For any  $t \in [0, 1]$ , let  $(v_n(t))_n$  be a representative of  $V(t)$ . Define a path of unitaries  $U : [0, 1] \rightarrow F(\mathcal{W})$  by

$$U(t) := w^*[(h_n \otimes v_n(t))_n]w$$

for any  $t \in [0, 1]$ . Then  $U$  has the desired property.  $\square$

The following theorem is the main theorem in this section.

**Theorem 5.7.** Let  $u$  be a unitary element in  $F(\mathcal{W})$ . For any  $\varepsilon > 0$ , there exists a path of unitaries  $U : [0, 1] \rightarrow F(\mathcal{W})$  such that

$$U(0) = 1, \quad U(1) = u, \quad \text{Lip}(U) < 2\pi + \varepsilon.$$

*Proof.* Let  $\delta > 0$ . We denote by  $\mu$  the probability measure on  $\mathbb{T}$  corresponding to  $f \mapsto \tau_\omega(f(u))$ . Since  $\mu(\mathbb{T}) = 1$ , there exists an element  $z_0$  in  $\mathbb{T}$  such that  $\mu(\{z \in \mathbb{T} \mid |z - z_0| < \delta\}) > 0$ . Let  $h$  be a positive element in  $C(\mathbb{T})$  such that

$$\{z \in \mathbb{T} \mid |z - z_0| < \delta\} \subset \overline{\text{supp } h} \subset \{z \in \mathbb{T} \mid |z - z_0| < 2\delta\}.$$

Then we have  $d_{\tau_\omega}(h(u)) > 0$ . Proposition 4.2 implies that there exists projection  $p$  in  $\overline{h(u)F(\mathcal{W})h(u)}$  such that  $\tau_\omega(p) > 0$ . Similar arguments as in the proof of [6, Lemma 1.7] show that there exist a unitary element  $u'$  in  $(1 - p)F(\mathcal{W})(1 - p)$  and a path of unitaries  $V_1 : [0, 1] \rightarrow F(\mathcal{W})$  such that

$$V_1(0) = u' + z_0 p, \quad V_1(1) = u, \quad \text{Lip}(V_1) < \varepsilon.$$

Indeed, we have

$$\|u - ((1 - p)u(1 - p) + z_0 p)\| < 6\delta$$

and, taking a sufficiently small  $\delta > 0$  and using polar decomposition, we obtain a path of unitaries  $V_1$  as above. By the proof of Lemma 5.5, it is easy to see that there exist a unitary element  $v$  in  $F(\mathcal{W})$  such that  $\tau_\omega(f(v)) > 0$  for any  $f \in C(\mathbb{T})_+ \setminus \{0\}$ . Using Lemma 5.6, Lemma 3.6 and the slow reindexation trick, we may assume that  $vp = pv$  and  $\tau_\omega(pv) = \tau_\omega(p)\tau_\omega(v)$ , and we see that there exists a path of unitaries  $V_2 : [0, 1] \rightarrow F(\mathcal{W})$  such that

$$V_2(0) = u' + vp, \quad V_2(1) = u' + z_0 p, \quad \text{Lip}(V_2) \leq \pi.$$

(See, for example, [42] for the slow reindexation trick.) Since we have  $\tau_\omega(f(u' + vp)) = \tau_\omega(f(u')) + \tau_\omega(f(v))\tau_\omega(p) > 0$  for any  $f \in C(\mathbb{T})_+ \setminus \{0\}$ , it follows from Lemma 5.6 that there exists a path of unitaries  $V_3 : [0, 1] \rightarrow F(\mathcal{W})$  such that

$$V_3(0) = 1, \quad V_3(1) = u' + vp, \quad \text{Lip}(V_3) \leq \pi.$$

Connecting  $V_1$ ,  $V_2$  and  $V_3$ , we obtain a desired path.  $\square$

## 6. ROHLIN TYPE THEOREM

In this section we shall show that every trace scaling automorphism of  $\mathcal{W} \otimes \mathbb{K}$  has the Rohlin property.

**Definition 6.1.** Let  $A$  be a  $\sigma$ -unital  $C^*$ -algebra, and let  $\alpha$  be an automorphism of  $A$ . We say that  $\alpha$  has the *Rohlin property* if for any  $k \in \mathbb{N}$ , there exist projections  $\{p_{1,i}\}_{i=0}^{k-1}$  and  $\{p_{2,j}\}_{j=0}^k$  in  $F(A)$  such that

$$\sum_{i=0}^{k-1} p_{1,i} + \sum_{j=0}^k p_{2,j} = 1, \quad \alpha(p_{1,i}) = p_{1,i+1}, \quad \alpha(p_{2,j}) = p_{2,j+1}$$

for any  $i = 0, 1, \dots, k-2$  and  $j = 0, 1, \dots, k-1$ .

If  $A$  is unital, then the definition above coincides with the usual definition (see, for example, [26]).

We identify  $F(\mathcal{W} \otimes \mathbb{K})$  with  $F(\mathcal{W})$ . We denote by the same symbol  $\tau_\omega$  the unique tracial state on  $F(\mathcal{W} \otimes \mathbb{K})$  for simplicity. Note that for any  $[(x_n)_n] \in F(\mathcal{W} \otimes \mathbb{K})$ ,  $\tau_\omega([(x_n)_n]) = \lim_{n \rightarrow \omega} \tau \otimes \text{Tr}(x_n h) / \tau \otimes \text{Tr}(h)$  for some  $h \in \text{Ped}(\mathcal{W} \otimes \mathbb{K})_+ \setminus \{0\}$  (see Remark 2.4). The following lemma is a variant of [36, Theorem 3.4].

**Lemma 6.2.** Let  $\alpha$  be a trace scaling automorphism of  $\mathcal{W} \otimes \mathbb{K}$ . Then for any  $k \in \mathbb{N}$ , there exists a positive contraction  $f$  in  $F(\mathcal{W} \otimes \mathbb{K})$  such that

$$\tau_\omega(f) = \frac{1}{k}, \quad f\alpha^j(f) = 0$$

for any  $j = 1, 2, \dots, k-1$ .

*Proof.* Note that  $\pi_{\tau \otimes \text{Tr}}(\mathcal{W} \otimes \mathbb{K})''$  is the AFD factor of type  $\text{II}_\infty$  and  $\tilde{\alpha}$  is a trace scaling automorphism. Hence it follows from [5, Lemma 5] and [5, Theorem 1.2.5] that there exist projections  $\{\tilde{p}_j\}_{j=1}^k$  in  $(\pi_{\tau \otimes \text{Tr}}(\mathcal{W} \otimes \mathbb{K})'')_\omega$  such that

$$\sum_{j=1}^k \tilde{p}_j = 1, \quad \tilde{\alpha}(\tilde{p}_j) = \tilde{p}_{j+1}$$

for any  $j = 1, 2, \dots, k-1$ . Proposition 2.3 implies that there exists a positive contraction  $e$  in  $F(\mathcal{W} \otimes \mathbb{K})$  such that  $\varrho_{\mathcal{W} \otimes \mathbb{K}}(e) = \tilde{p}_1$ . It is easy to see that  $\tau_\omega(e) = 1/k$ . Let  $(e_n)_n$  be a representative of  $e$ . Then

$$\|\pi_{\tau \otimes \text{Tr}}(e_n \alpha^j(e_n))\hat{h}\|_2 \rightarrow 0$$

as  $n \rightarrow \omega$  for any  $h \in \text{Ped}(\mathcal{W} \otimes \mathbb{K})$  and  $j = 1, 2, \dots, k-1$ . By similar arguments as in the proof of [36, Proposition 3.3], one can prove the lemma. Indeed, put

$$e'_n := e_n^{\frac{1}{2}} \left( \sum_{j=1}^{k-1} \alpha^j(e_n) \right) e_n^{\frac{1}{2}}.$$

Then  $\|\pi_{\tau \otimes \text{Tr}}(e'_n)\hat{h}\|_2 \rightarrow 0$  as  $n \rightarrow \omega$  for any  $h \in \text{Ped}(\mathcal{W} \otimes \mathbb{K})$ . For any  $\varepsilon > 0$ , define

$$g_\varepsilon(t) := \begin{cases} \varepsilon^{-1}t & \text{if } t \in [0, \varepsilon] \\ 1 & \text{if } t \in [\varepsilon, \infty) \end{cases}$$

and let  $f_n := e_n - e_n^{\frac{1}{2}} g_\varepsilon(e'_n) e_n^{\frac{1}{2}}$ . The same proof as [36, Proposition 3.3] shows that

$$\|f_n \alpha^j(f_n)\|^2 < \varepsilon$$

for any  $j = 1, 2, \dots, k-1$ . For  $h \in \text{Ped}(\mathcal{W} \otimes \mathbb{K})_{+,1}$ , we have

$$\begin{aligned} \lim_{n \rightarrow \omega} \|\pi_{\tau \otimes \text{Tr}}(f_n - e_n)\hat{h}\|_2 &= \lim_{n \rightarrow \omega} \|\pi_{\tau \otimes \text{Tr}}(e_n^{\frac{1}{2}}g_\varepsilon(e'_n)e_n^{\frac{1}{2}})\hat{h}\|_2 \\ &\leq \lim_{n \rightarrow \omega} \|\pi_{\tau \otimes \text{Tr}}(g_\varepsilon(e'_n)e_n^{\frac{1}{2}})\hat{h}\|_2 \\ &= \lim_{n \rightarrow \omega} \|\pi_{\tau \otimes \text{Tr}}(g_\varepsilon(e'_n)h^{\frac{1}{2}}e_n^{\frac{1}{2}})\widehat{h^{\frac{1}{2}}}\|_2 \\ &\leq \lim_{n \rightarrow \omega} \|\pi_{\tau \otimes \text{Tr}}(g_\varepsilon(e'_n))\widehat{h^{\frac{1}{2}}}\|_2 = 0. \end{aligned}$$

Hence  $\tau_\omega([(e_n)_n]) = \tau_\omega([(f_n)_n])$  (see Remark 2.4). Therefore we obtain the conclusion by the usual diagonal argument.  $\square$

**Lemma 6.3.** Let  $\alpha$  be a trace scaling automorphism of  $\mathcal{W} \otimes \mathbb{K}$ . Then for any  $k \in \mathbb{N}$ , there exists a projection  $p$  in  $F(\mathcal{W} \otimes \mathbb{K})$  such that

$$\tau_\omega(p) = \frac{1}{k}, \quad p\alpha^j(p) = 0$$

for any  $j = 1, 2, \dots, k-1$ , and  $p$  is Murray-von Neumann equivalent to  $\alpha(p)$

*Proof.* We identify  $F(\mathcal{W} \otimes \mathbb{K})$  with  $F(\mathcal{W} \otimes \mathbb{K} \otimes M_{2^\infty})$ . Let  $\{h_n\}_{n \in \mathbb{N}}$  be an approximate unit for  $\mathcal{W} \otimes \mathbb{K}$ . By Lemma 6.2, there exists a positive contraction  $f$  in  $F(\mathcal{W} \otimes \mathbb{K} \otimes M_{2^\infty})$  such that

$$\tau_\omega(f) = \frac{1}{k}, \quad f\alpha^j(f) = 0$$

for any  $j = 1, 2, \dots, k-1$ . Since  $f$  is a contraction,  $d_{\tau_\omega}(f) \geq \tau_\omega(f) = 1/k$ . Let  $\varepsilon > 0$ . There exists a projection  $(q_n)_n$  in  $(M_{2^\infty})_\omega$  such that  $\tau'_\omega((q_n)_n) = 1/k - \varepsilon$ , where  $\tau'$  is the unique tracial state on  $M_{2^\infty}$ . Proposition 4.1 implies that there exists an element  $r$  such that  $r^*fr = [(h_n \otimes q_n)_n]$ . Put  $p = f^{\frac{1}{2}}rr^*f^{\frac{1}{2}}$ , then

$$\tau_\omega(p) = \frac{1}{k} - \varepsilon, \quad p\alpha^j(p) = 0$$

for any  $j = 1, 2, \dots, k-1$ , and  $p$  is Murray-von Neumann equivalent to  $[(h_n \otimes q_n)_n]$ . Define an automorphism  $\beta$  of  $\mathcal{W} \otimes \mathbb{K} \otimes M_{2^\infty}$  by  $\beta := \alpha \otimes \text{id}_{M_{2^\infty}}$ . It is easy to see that  $[(h_n \otimes q_n)_n]$  is Murray-von Neumann equivalent to  $\beta([(h_n \otimes q_n)_n])$ . By Theorem 2.7,  $\alpha$  is approximately unitarily equivalent  $\beta$ . (Note that we regard here  $\alpha$  as an automorphism of  $\mathcal{W} \otimes \mathbb{K} \otimes M_{2^\infty}$ .) Hence it follows from [34, Lemma 4.3] that  $\beta([(h_n \otimes q_n)_n])$  is unitarily equivalent to  $\alpha([(h_n \otimes q_n)_n])$ . Since  $\alpha([(h_n \otimes q_n)_n])$  is Murray-von Neumann equivalent to  $\alpha(p)$ ,  $p$  is Murray-von Neumann equivalent to  $\alpha(p)$ . Therefore we obtain the conclusion by the usual diagonal argument.  $\square$

The following theorem is the main theorem in this section. The proof is based on [24] and [25].

**Theorem 6.4.** Let  $\alpha$  be a trace scaling automorphism of  $\mathcal{W} \otimes \mathbb{K}$ . Then  $\alpha$  has the Rohlin property.

*Proof.* For any  $N \geq 2$ , it follows from Lemma 6.3 that there exists a (non-unital) homomorphism  $\varphi$  from  $M_N(\mathbb{C})$  to  $F(\mathcal{W} \otimes \mathbb{K})$  such that

$$\alpha(\varphi(e_{ij})) = \varphi(e_{i+1,j+1})$$

for any  $i, j = 1, \dots, N-1$ , where  $\{e_{ij}\}_{i,j=1}^N$  is the standard matrix units of  $M_N(\mathbb{C})$  (see [24, Lemma 4.3] for details). By the same argument as in the proof of [24, Lemma 2.1] and the usual diagonal argument, we see that for any  $m \in \mathbb{N}$ , there exists a projection  $p$  in  $F(\mathcal{W} \otimes \mathbb{K})$  such that

$$\tau_\omega(p) = \frac{1}{m}, \quad p\alpha^j(p) = 0, \quad \alpha^m(p) = p$$

for any  $j = 1, 2, \dots, m-1$ . Note that there exists an element  $v$  in  $F(\mathcal{W} \otimes \mathbb{K})$  such that

$$v^*v = 1 - \sum_{j=0}^{m-1} \alpha^j(p), \quad vv^* \leq p$$

because  $\mathcal{W}$  has property (SI) and we have  $\tau_\omega(1 - \sum_{j=0}^{m-1} \alpha^j(p)) = 0$ . Therefore the rest of the proof is the same as [24, Theorem 2.1] and [25, Lemma 4.4].  $\square$

An automorphism  $\alpha$  of  $\mathcal{W}$  is said to be *strongly outer* if  $\tilde{\alpha}$  is not inner in  $\pi_\tau(\mathcal{W})''$ . The same proof as Lemma 6.2 shows the following lemma.

**Lemma 6.5.** Let  $\alpha$  be an automorphism of  $\mathcal{W}$  such that  $\alpha^m$  is strongly outer for any  $m \in \mathbb{Z} \setminus \{0\}$ . Then for any  $k \in \mathbb{N}$ , there exists a positive contraction  $f$  in  $F(\mathcal{W} \otimes \mathbb{K})$  such that

$$\tau_\omega(f) = \frac{1}{k}, \quad f\alpha^j(f) = 0$$

for any  $j = 1, 2, \dots, k-1$ .

**Lemma 6.6.** Let  $\alpha$  be an automorphism of  $\mathcal{W}$  such that  $\alpha^m$  is strongly outer for any  $m \in \mathbb{Z} \setminus \{0\}$ . Then for any  $k \in \mathbb{N}$ , there exists a projection  $p$  in  $F(\mathcal{W})$  such that

$$\tau_\omega(p) = \frac{1}{k}, \quad p\alpha^j(p) = 0$$

for any  $j = 1, 2, \dots, k-1$ , and  $p$  is Murray-von Neumann equivalent to  $\alpha(p)$ .

*Proof.* In a similar way as in Lemma 6.3, we see that there exists a projection  $p$  in  $F(\mathcal{W} \otimes \mathbb{K})$  such that

$$\tau_\omega(p) = \frac{1}{k}, \quad p\alpha^j(p) = 0$$

for any  $j = 1, 2, \dots, k-1$ . Since every automorphism of  $\mathcal{W}$  is approximately inner by Theorem 2.7, we obtain the conclusion by [34, Lemma 4.3].  $\square$

By the same arguments as in the proof of Theorem 6.7, we obtain the following theorem.

**Theorem 6.7.** Let  $\alpha$  be an automorphism of  $\mathcal{W}$  such that  $\alpha^m$  is strongly outer for any  $m \in \mathbb{Z} \setminus \{0\}$ . Then  $\alpha$  has the Rohlin property.

## 7. OUTER CONJUGACY

In this section we shall classify trace scaling automorphisms of  $\mathcal{W} \otimes \mathbb{K}$  up to outer conjugacy. Using Theorem 5.7 instead of [16, Lemma 1], we can prove the following theorem by essentially the same argument as in the proof of [16, Theorem 1]. (See also [17] and [28].)

**Theorem 7.1.** Let  $A$  be a C\*-algebra which is isomorphic to  $\mathcal{W}$  or  $\mathcal{W} \otimes \mathbb{K}$ , and let  $\alpha$  be an automorphism of  $A$  with the Rohlin property. For any unitary element  $u$  in  $F(A)$ , there exists a unitary element  $v$  in  $F(A)$  such that  $u = v\alpha(v)^*$ .

The following lemma is an immediate consequence of the theorem above and Corollary 4.10.

**Lemma 7.2.** Let  $A$  be a C\*-algebra which is isomorphic to  $\mathcal{W}$  or  $\mathcal{W} \otimes \mathbb{K}$ , and let  $\alpha$  be an automorphism of  $A$  with the Rohlin property. Then for any finite subsets  $E \subset A^\sim$ ,  $F \subset A$  and  $\varepsilon > 0$ , there exist a finite subset  $G \subset A$  and  $\delta > 0$  such that the following holds. If  $u$  is a unitary element in  $M(A)$  satisfying

$$\|[u, a]\| < \delta$$

for any  $a \in G$ , then there exists a unitary element  $v$  in  $A^\sim$  such that

$$\|[v, x]\| < \varepsilon, \quad \|(u - v\alpha(v)^*)y\| < \varepsilon, \quad \|y(u - v\alpha(v)^*)\| < \varepsilon$$

for any  $x \in E$  and  $y \in F$ .

The following theorem is the main theorem in this paper.

**Theorem 7.3.** Let  $\alpha$  and  $\beta$  be trace scaling automorphisms of  $\mathcal{W} \otimes \mathbb{K}$ . Then  $\alpha$  and  $\beta$  are outer conjugate if and only if  $\lambda(\alpha) = \lambda(\beta)$ .

*Proof.* The only if part is obvious. We will show the if part. Theorem 6.4 implies that  $\alpha$  and  $\beta$  have the Rohlin property. Since  $\lambda(\alpha) = \lambda(\beta)$ ,  $\alpha$  is approximately unitarily equivalent to  $\beta$  by Theorem 2.7. Therefore we obtain the conclusion by Lemma 7.2 and the Bratteli-Elliott-Evans-Kishimoto intertwining argument [13] (see also [18], [26], [33], [40] and [49] for similar arguments). Indeed, let  $\{x_n\}_{n \in \mathbb{N}}$  be a dense set in the unit ball of  $\mathcal{W} \otimes \mathbb{K}$ . By induction, we shall construct sequences of automorphisms  $\{\alpha_{2n}\}_{n=0}^\infty, \{\beta_{2n+1}\}_{n=0}^\infty$  of  $\mathcal{W} \otimes \mathbb{K}$  and sequences of unitaries  $\{u_n\}_{n=0}^\infty, \{v_n\}_{n=0}^\infty, \{w_n\}_{n=0}^\infty, \{\tilde{w}_n\}_{n=0}^\infty$  in  $(\mathcal{W} \otimes \mathbb{K})^\sim$  as follows: Put  $\alpha_0 := \alpha, \beta_1 := \beta$ , and let  $F_1 := \{x_1, x_1^*\}, F'_1 := \{x_1, x_1^*\}, E_1 := \{1\}$ . Applying Lemma 7.2 to  $\beta_1, E_1, F'_1$  and  $1/2$ , we obtain a finite subset  $G_1 \subset \mathcal{W} \otimes \mathbb{K}$  and  $\delta_1 > 0$ . Set

$$F_2 := \beta_1^{-1}(G_1) \cup F_1 \cup \{x_2, x_2^*\}.$$

Since  $\beta_1$  is approximately unitarily equivalent to  $\alpha_0$ , there exists a unitary element  $u_0$  in  $(\mathcal{W} \otimes \mathbb{K})^\sim$  such that

$$\|\beta_1(a) - u_0\alpha_0(a)u_0^*\| < \frac{\delta_1}{2} \quad (1)$$

for any  $a \in F_2$ . Put  $\alpha_2 := \text{Ad}(u_0) \circ \alpha_0, v_0 := u_0$ , and let  $w_0 := u_0\alpha_0(v_0)v_0^*, \tilde{w}_0 := w_0$ . Set

$$E_2 := F_2 \cup \text{Ad}(v_0)(F_2) \cup \{\tilde{w}_0\}, \quad F'_2 := F_2 \cup F_2\tilde{w}_0^*.$$

Applying Lemma 7.2 to  $\alpha_2, E_2, F'_2$  and  $1/2^2$ , we obtain a finite subset  $G_2 \subset \mathcal{W} \otimes \mathbb{K}$  and  $\delta_2 > 0$ . We may assume that  $\delta_2 < \delta_1/2$ . Set

$$F_3 := \alpha_2^{-1}(G_2) \cup F_2 \cup \{x_3, x_3^*\}.$$

Since  $\alpha_2$  is approximately unitarily equivalent to  $\beta_1$ , there exists a unitary element  $u_1$  in  $(\mathcal{W} \otimes \mathbb{K})^\sim$  such that

$$\|\alpha_2(a) - u_1\beta_1(a)u_1^*\| < \frac{\delta_2}{2} \quad (2)$$

for any  $a \in F_2$ . Put  $\beta_3 := \text{Ad}(u_1) \circ \beta_1$ . By (1) and (2), we have

$$\|[u_1, a]\| < \delta_1$$

for any  $a \in G_1$ . Hence there exists a unitary element  $v_1$  such that

$$\|[v_1, x]\| < \frac{1}{2}, \quad \|y(u_1 - v_1\beta_1(v_1)^*)\| < \frac{1}{2}$$

for any  $x \in E_1$  and  $y \in F'_1$  by Lemma 7.2. Put  $w_1 := u_1\beta_1(v_1)v_1^*, \tilde{w}_1 := w_1$ , and set

$$E_3 := F_3 \cup \text{Ad}(v_1)(F_3) \cup \{\tilde{w}_1\}, \quad F'_3 := F_3 \cup F_3\tilde{w}_1^*.$$

Applying Lemma 7.2 to  $\beta_3, E_3, F'_3$  and  $1/2^3$ , we obtain a finite subset  $G_3 \subset \mathcal{W} \otimes \mathbb{K}$  and  $\delta_3 > 0$ . We may assume that  $\delta_3 < \delta_2/2$ . Set

$$F_4 := \beta_3^{-1}(G_3) \cup F_3 \cup \{x_4, x_4^*\}.$$

Since  $\beta_3$  is approximately unitarily equivalent to  $\alpha_2$ , there exists a unitary element  $u_2$  in  $(\mathcal{W} \otimes \mathbb{K})^\sim$  such that

$$\|\beta_3(a) - u_2\alpha_2(a)u_2^*\| < \frac{\delta_3}{2} \quad (3)$$



for any  $a \in F_3$ . Put  $\alpha_4 := \text{Ad}(u_2) \circ \alpha_2$ . By (2) and (3), we have

$$\|[u_2, a]\| < \delta_2$$

for any  $a \in G_2$ . Hence there exists a unitary element  $v_2$  such that

$$\|[v_2, x]\| < \frac{1}{2^2}, \quad \|y(u_2 - v_2 \alpha_2(v_2)^*)\| < \frac{1}{2^2}$$

for any  $x \in E_2$  and  $y \in F'_2$  by Lemma 7.2. Put  $w_2 := u_2 \alpha_2(v_2) v_2^*$ , and let  $\tilde{w}_2 := w_2 v_2 \tilde{w}_0 v_2^*$ . Set

$$E_4 := F_4 \cup \text{Ad}(v_2 v_0)(F_4) \cup \{\tilde{w}_2\}, \quad F'_4 := F_4 \cup F_4 \tilde{w}_2^*.$$

Applying Lemma 7.2 to  $\alpha_4$ ,  $E_4$ ,  $F'_4$  and  $1/2^4$ , we obtain a finite subset  $G_4 \subset \mathcal{W} \otimes \mathbb{K}$  and  $\delta_4 > 0$ . We may assume that  $\delta_4 < \delta_3/2$ . Set

$$F_5 := \alpha_4^{-1}(G_4) \cup F_4 \cup \{x_5, x_5^*\}.$$

Since  $\alpha_4$  is approximately unitarily equivalent to  $\beta_3$ , there exists a unitary element  $u_3$  in  $(\mathcal{W} \otimes \mathbb{K})^\sim$  such that

$$\|\alpha_4(a) - u_3 \beta_3(a) u_3^*\| < \frac{\delta_4}{2} \quad (4)$$

for any  $a \in F_4$ . Put  $\beta_5 := \text{Ad}(u_3) \circ \beta_3$ . By (3) and (4), we have

$$\|[u_3, a]\| < \delta_3$$

for any  $a \in G_3$ . Hence there exists a unitary element  $v_3$  such that

$$\|[v_3, x]\| < \frac{1}{2^3}, \quad \|y(u_3 - v_3 \beta_3(v_3)^*)\| < \frac{1}{2^3}$$

for any  $x \in E_3$  and  $y \in F'_3$  by Lemma 7.2. Put  $w_3 := u_3 \beta_3(v_3) v_3^*$ , and let  $\tilde{w}_3 := w_3 v_3 \tilde{w}_1 v_3^*$ .

Repeating this process, we obtain sequences  $\{\alpha_{2n}\}_{n=0}^\infty$ ,  $\{\beta_{2n+1}\}_{n=0}^\infty$ ,  $\{u_n\}_{n=0}^\infty$ ,  $\{v_n\}_{n=0}^\infty$ ,  $\{w_n\}_{n=0}^\infty$  and  $\{\tilde{w}_n\}_{n=0}^\infty$  such that

$$(i) \quad \alpha_{2n} = \text{Ad}(u_{2n-2}) \circ \alpha_{2n-2}, \quad \beta_{2n+1} = \text{Ad}(u_{2n-1}) \circ \beta_{2n-1},$$

$$(ii) \quad w_{2n} = u_{2n} \alpha_{2n}(v_{2n}) v_{2n}^*, \quad w_{2n+1} = u_{2n+1} \beta_{2n+1}(v_{2n+1}) v_{2n+1}^*,$$

$$\tilde{w}_{n+1} = w_{n+1} v_{n+1} \tilde{w}_{n-1} v_{n+1}^*,$$

$$(iii) \quad \|\beta_{2n-1}(x_i) - \alpha_{2n}(x_i)\| < \frac{\delta_{2n-1}}{2}, \quad 1 \leq \forall i \leq 2n,$$

$$(iv) \quad \|\alpha_{2n}(x_i) - \beta_{2n+1}(x_i)\| < \frac{\delta_{2n}}{2}, \quad 1 \leq \forall i \leq 2n+1,$$

$$(v) \quad \|[v_{2n}, x_i]\| < \frac{1}{2^{2n}}, \quad \|[v_{2n}, \text{Ad}(v_{2n-2} v_{2n-4} \cdots v_0)(x_i)]\| < \frac{1}{2^{2n}},$$

$$\|[v_{2n}, \tilde{w}_{2n-2}]\| < \frac{1}{2^{2n}}, \quad 1 \leq \forall i \leq 2n,$$

$$(vi) \quad \|[v_{2n+1}, x_i]\| < \frac{1}{2^{2n+1}}, \quad \|[v_{2n+1}, \text{Ad}(v_{2n-1} v_{2n-3} \cdots v_1)(x_i)]\| < \frac{1}{2^{2n+1}},$$

$$\|[v_{2n+1}, \tilde{w}_{2n-1}]\| < \frac{1}{2^{2n+1}}, \quad 1 \leq \forall i \leq 2n+1,$$

$$(vii) \quad \|x_i(u_{2n} - v_{2n} \alpha_{2n}(v_{2n})^*)\| < \frac{1}{2^{2n}},$$

$$\|x_i^* \tilde{w}_{2n-2}^*(u_{2n} - v_{2n} \alpha_{2n}(v_{2n})^*)\| < \frac{1}{2^{2n}}, \quad 1 \leq \forall i \leq 2n,$$

$$(viii) \quad \|x_i(u_{2n+1} - v_{2n+1} \beta_{2n+1}(v_{2n+1})^*)\| < \frac{1}{2^{2n+1}},$$

$$\|x_i^* \tilde{w}_{2n-1}^*(u_{2n+1} - v_{2n+1} \beta_{2n+1}(v_{2n+1})^*)\| < \frac{1}{2^{2n+1}}, \quad 1 \leq \forall i \leq 2n+1,$$

for any  $n \in \mathbb{N}$ , where  $\{\delta_n\}_{n \in \mathbb{N}}$  is a sequence of positive numbers such that  $\delta_n < \delta_{n-1}/2$ .

For any  $n \in \mathbb{N}$ , define  $\theta_n := \text{Ad}(v_{2n}v_{2n-2} \cdots v_0)$  and  $\gamma_n := \text{Ad}(v_{2n+1}v_{2n-1} \cdots v_1)$ . By (v) and (vi), the same proof as [18, Theorem 3.5] shows that there exist automorphisms  $\theta$  and  $\gamma$  of  $\mathcal{W} \otimes \mathbb{K}$  such that

$$\theta(x) = \lim_{n \rightarrow \infty} \theta_n(x), \quad \gamma(x) = \lim_{n \rightarrow \infty} \gamma_n(x)$$

for any  $x \in \mathcal{W} \otimes \mathbb{K}$ .

For any  $n \in \mathbb{N}$ , (ii) and (vii) imply that

$$\|x_i(w_{2n} - 1)\| < \frac{1}{2^{2n}}, \quad \|(w_{2n} - 1)\tilde{w}_{2n-2}x_i\| < \frac{1}{2^{2n}}$$

for any  $i = 1, 2, \dots, 2n$ . By (ii) and (v), we have for any  $n \in \mathbb{N}$ ,

$$\begin{aligned} \|x_i(\tilde{w}_{2n} - \tilde{w}_{2n-2})\| &= \|x_i w_{2n} v_{2n} \tilde{w}_{2n-2} v_{2n}^* - x_i \tilde{w}_{2n-2} v_{2n} v_{2n}^*\| \\ &\leq \|x_i w_{2n} [v_{2n}, \tilde{w}_{2n-2}]\| + \|x_i (w_{2n} - 1) \tilde{w}_{2n-2} v_{2n}\| \\ &< \frac{1}{2^{2n-1}} \end{aligned}$$

and

$$\begin{aligned} \|(\tilde{w}_{2n} - \tilde{w}_{2n-2})x_i\| &= \|w_{2n} v_{2n} \tilde{w}_{2n-2} v_{2n}^* x_i - \tilde{w}_{2n-2} v_{2n} v_{2n}^* x_i\| \\ &\leq \|w_{2n} [v_{2n}, \tilde{w}_{2n-2}] v_{2n}^* x_i\| + \|(w_{2n} - 1) \tilde{w}_{2n-2} x_i\| \\ &< \frac{1}{2^{2n-1}} \end{aligned}$$

for any  $i = 1, 2, \dots, 2n$ . Therefore  $\{\tilde{w}_{2n}\}_{n \in \mathbb{N}}$  is a strict Cauchy sequence of unitaries in  $(\mathcal{W} \otimes \mathbb{K})^\sim$ . Since  $M(\mathcal{W} \otimes \mathbb{K})$  is strictly complete, there exists a unitary element  $w'_0$  in  $M(\mathcal{W} \otimes \mathbb{K})$  such that  $\{\tilde{w}_{2n}\}_{n \in \mathbb{N}}$  converges strictly to  $w'_0$ . In a similar way, we see that there exists a unitary element  $w'_1$  in  $M(\mathcal{W} \otimes \mathbb{K})$  such that  $\{\tilde{w}_{2n+1}\}_{n \in \mathbb{N}}$  converges strictly to  $w'_1$ .

It can be easily checked that

$$\alpha_{2n+2} = \text{Ad}(\tilde{w}_{2n}) \circ \theta_n \circ \alpha \circ \theta_n^{-1}, \quad \beta_{2n+3} = \text{Ad}(\tilde{w}_{2n+1}) \circ \gamma_n \circ \beta \circ \gamma_n^{-1}$$

for any  $n \in \mathbb{N}$ . It follows from (iv) that for any  $n \in \mathbb{N}$ , we have

$$\|\alpha_{2n+2}(x_i) - \beta_{2n+3}(x_i)\| < \frac{\delta_{2n+2}}{2}$$

for any  $i = 1, 2, \dots, 2n+3$ . Therefore we see that

$$\text{Ad}(w'_0) \circ \theta \circ \alpha \circ \theta^{-1}(x) = \text{Ad}(w'_1) \circ \gamma \circ \beta \circ \gamma^{-1}(x)$$

for any  $x \in \mathcal{W} \otimes \mathbb{K}$  because  $\{\tilde{w}_n\}_{n \in \mathbb{N}}$  is a bounded sequence and  $\lim_{n \rightarrow \infty} \delta_n = 0$ .  $\square$

By Theorem 6.7 and the same proof as above, we obtain the following theorem.

**Theorem 7.4.** Let  $\alpha$  and  $\beta$  be automorphisms of  $\mathcal{W}$ . If  $\alpha^m$  and  $\beta^m$  are strongly outer for any  $m \in \mathbb{Z} \setminus \{0\}$ , then  $\alpha$  and  $\beta$  are outer conjugate.

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